THE CONCEPTION OF THE INFINITE

FULLERTON

THE CONCEPTION OF THE INFINITE, AND THE SOLUTION OF THE MATHEMATICAL ANTINOMIES:

A STUDY IN PSYCHOLOGICAL ANALYSIS.

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PREFACE.

The question treated in this little volume is one of no small interest from several quite different points of view. To one interested in lucid and systematic thinking, the tangle of thought which has always obtained in this corner of the philosophic field cannot but be repulsive and irritating. To be told that of two impossible things one must be true; that of the same two lines one may be looked upon as, at pleasure, equal to, less than, or greater than the other, both remaining unchanged; that Achilles, running rapidly, can never overtake the tortoise, moving slowly; to be told all this seriously, by men whose calling it is to think and to teach others to think, is well calculated to bring not merely suspicion but contempt upon speculative thought, and deservedly. Who has not puzzled, on his first

introduction to Logic, over some of these antinomies, and been silenced unconvinced by the practical demonstration,—as by walking, in the case of the argument against motion,—which cuts the knot but does not solve it, leaving in the mind a disagreeable sense that the argument must be wrong somewhere, and yet a consciousness that it certainly seems perfectly sound? When the metaphysician proves to us that a rhinoceros is a mosquito, his chain of reasoning is rendered innocuous by the striking incongruity of the conclusion; but if we observe no flaw in his reasoning, we cannot help recognizing the perplexing truth that it is the experienced fact alone which has prevented assent, and that a precisely similar argument, the conclusion of which cannot be similarly tested, may yet induce assent, though equally erroneous. If we have no better reason for rejecting an argument, what can be our criterion when we leave the sphere of the immediately palpable? He who has convinced himself that the minute hand of a clock cannot overtake the hour hand, will be enlightened when the clock strikes at noon; but he who has followed Mr. Spencer into his discussions regarding our no-

tions of infinite space or time, will be filled with inward dismay if he hang his hope upon any such practical expedient. Civil history cannot be studied in the laboratory, nor erroneous ideas as to infinite space rectified with the aid of the foot-rule. In this sphere, too, the question of a careful and thorough analysis of our conception of the infinite is of more than a merely intellectual interest, and any erroneous conception which can blossom out into such a development as the "Philosophy of the Conditioned," with its implications, has a religious significance which cannot be overlooked. The analysis of this single conception is, moreover, of importance as throwing light upon the procedure of thought in general, and will to many be of more interest in this connection than for its own sake. I have endeavored to write with extreme clearness and simplicity, and to avoid, as much as possible, all issues not directly connected with the immediate subject; and whether my discussion meet with assent or dissent, I do not think it will be charged with the obscurity characteristic of discussions upon this much-mooted topic.

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CHAPTER I.

INTRODUCTORY.

THE doctrine that there may be an infinity of worlds, thought Plutarch,* is to be repudiated. Providence could not possibly take charge of so many; "troublesome and boundless infinity" could be grasped by no consciousness.

Plutarch's decision as to the unknowability of the infinite the student of the history of thought will find reiterated by thinkers who agree upon little else than the one point, that

* "De defectu oraculorum," c. 24.

when we leave the finite and talk of the infinite we are playing with a word,—deceiving ourselves into believing that we can know what is in its very nature inconceivable.

The notion of the infinite, it is said,* is a negative one: all our experience of objects being of them as finite, we can think to ourselves the negation of this condition, and thus form a negative conception,—*i.e.*, not an affirmation of a quality or attribute, but a simple denial of a quality known. But to know an object—for example, space—as infinite, this is beyond our power. The human mind is finite; all which can become an object of consciousness is finite; and "troublesome and boundless infinity" cannot be an object of thought.

"Whereas, for this very reason," said Cudworth,† "because more could be added to the magnitude of the corporeal world infinitely, or without end, therefore it is impossible, that it should ever be positively and actually infinite;

* See Hamilton's Discussions, "Philosophy of the Unconditioned."

† "Intellectual System," chap. v., Andover, 1838, vol. ii. p. 45.

that is such as to which nothing more can possibly be added."

A favorite position: all attempts at cognizing the infinite must result in the indefinite; for the infinite can only be known by the progressive addition of finites, an addition which can be completed only in an infinite time; in other words, can never be completed. Each stage in the addition gives but the indefinite, and the last stage, the infinite, is, by the terms of the problem, unattainable.

How, indeed, it is asked, could a finite mind know the infinite? "Adequately to know what is infinite is to have infinite knowledge."* We cannot, surely, lay claim to that.

The famous antinomies of the philosopher of Königsberg have, more than anything else, stirred up the minds of men to a consideration of the problem. The indefinite, said Kant, we may know as a whole, not by passing successively over each of its parts, but immediately as a unit, by means of its limits; but in the case of the infinite, since there are no limits, we must arrive at our cognition by a successive

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* "The Battle of the Two Philosophies," p. 24.

addition of parts, which addition must necessarily be itself endless, and therefore the attempt thus to know the infinite futile. Although we are unable to conceive of an absolute commencement to time or of an absolute limit to space, we are not on that account justified in assuming either to be infinite, as this we can only know when we have passed successively over all spaces and all times. The whole question is one the decision of which is beyond the scope of human reason; both alternatives are equally inconceivable.

The doctrine of the inconceivability of the infinite is much dwelt upon by Sir William Hamilton as one phase of his cherished theory, the Philosophy of the Conditioned. He argues that space, for example, cannot be cognized as either infinitely extended and infinitely divisible, or, on the other hand, as absolutely limited, whether as a *maximum* or *minimum*, just as Kant argued. Mr. Mansel has accepted and defended* the positions taken by Hamilton; while the same arguments, which are used by Sir

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* See Mr. Mansel's "Philosophy of the Conditioned." London: 1866.

William Hamilton and his follower, Mr. Mansel, to elevate faith at the expense of reason or science, are to be found upon the pages of the "First Principles of Philosophy," where they are used by Mr. Spencer in support of a philosophy widely different from the Hamiltonian.

It would not be difficult to multiply testimonies to the inconceivability of the infinite, for the misconception which we find in Plutarch appears and reappears in divers forms in different ages and climes, much as the Wandering Jew may be supposed to have presented himself; and, to carry out the simile, the well-worn dress in which the doctrine usually comes to the surface—the statement that we can know the infinite only by an endless addition of finites—may be not inaptly compared to the threadbare garment on his back.

It would seem surprising that there could be so universal a misconception with regard to the nature of a conception actually present at some time or other in the mind probably of every man, and in the case of many not unfrequently present; a conception sufficiently familiar and important to be recognized and marked by its

appropriate name. But a very little knowledge of psychological processes will make one cognizant of the fact that the interpretation of consciousness is by no means the easy and simple task that by a novice it might be supposed to be.

There are but few who have an analytical knowledge of even the most common of their mental operations; and the average man is very literally incapable of telling what may at any moment be passing in his mind. It is a matter of surprise, for instance, to one unaccustomed to psychological analysis, to learn that *distance* is not directly perceived by vision, but that judgments of distance are the result of a rapid process of reasoning, and imply a generalization from past experience. The judgment seems instantaneous and intuitive. Those elements which are in fact visual and actually present are not distinguished from those present only by suggestion; that is, present in the imagination. The mental state is grasped as a unit, and for practical purposes no analysis into its elements is necessary. So it is with the concept, or general notion. That men form general notions, or at least represent in mind objects and their re-

lations in some way different from that in which simple intuitions are singly represented, all are ready to acknowledge. But that the psychical elements concerned in this act are not clearly apprehended is evident from the common warfare of Realism, Nominalism, and Conceptualism. So is it, again, in the case of memory. The act is usually described as if there were present in the mind the two elements of a present mental representation and an intuition, or presentation, its prototype, to which it is referred; while the picture before the mind is but one, as an examination of consciousness during the act will very readily show. And the same truth is illustrated in innumerable ways by the debates and disputes of philosophers, past and present, as to mind and the faculties in which it is manifested. As a typical instance may be given the analysis of consciousness left us by Mr. John Stuart Mill as compared with that presented in Sir William Hamilton's Lectures. Was not each describing what was present in his own mind? Whence the discrepancy?

Moreover, a clear apprehension of the constituent elements of a mental state does not

always seem to be necessary, from a practical point of view, to enable one to use that state as a unit, with substantial accuracy. The Nominalist, the Realist, and the Conceptualist all speak the same language, refer to the same objects, and, in the use of the complex mental phenomenon which they so variously analyze, are at one. A seaman, practised in the judgment of distances by a long training in his vocation, may be much more accurate in his judgments of distance than the accomplished author of the "New Theory of Vision" himself, though he may be quite ignorant of the mental process by which he arrives at his conclusions. The process itself is not directly affected by an analytic consciousness of the steps of the process, nor is its result. The mental state is in most cases recognized only as a unit, since it is as a unit that it is useful to the individual; and the name by which it is known expresses the general impression conveyed to the mind when the state is called up.

Now, it is manifest, that when one begins to analyze this vague and indefinitely grasped total, and to separate it into its elements, it is quite possible for him to confound some of the

' Berkeley

elements with elements somewhat similar, or to imagine the presence of elements closely connected by the laws of association with elements really present; in short, to find what is not there, and what, when in practice he uses the word indicating the state as a unit, he never means to express by it. And, in view of this fact, we may see how it is possible for the curious error regarding the inconceivability of the infinite, which has been adverted to, to have arisen, and to have held its place in the writings of philosophers. The word has always been used, and the ideas for which it stands have often been in men's minds; but in attempting to explicate the conception, we find that almost all place among the qualities it connotes a notion drawn from finites, and which is contradictory to the essential character of the conception. Few men have talked more about the infinite than Sir William Hamilton; and it is probable that every time he used the word it called up very much the same mental state in his mind as that which arose in the mind of Mr. John Stuart Mill when he read Sir William's Lectures; but the conception thus called up certainly did not contain the warring ele-

ments which Sir William finds in it when he undertakes to prove it unthinkable. And when Mr. Mill criticises Sir William Hamilton, and declares the infinite not inconceivable, he probably meant by the infinite just what Hamilton did; and yet when he tries to prove its conceivability, he finds in it what was not really contained in the mind of either when the word was used.

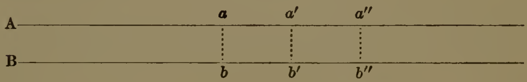
The problem is simply one of psychological analysis,—a question of what may be the true content of a complex mental state; and the fact that such an analysis may be incorrectly made need not surprise us, since erroneous analyses are only too common. What may well create surprise, however, is that the difficulties into which this erroneous analysis has led those guilty of it, the antinomies of which it is the source, have not called attention to the error, and that a rigorous analysis has not been employed to eliminate it. I will consider in the remaining chapters of this little book the errors which have arisen from a mistaken notion of the content of our conception of the infinite, and show that they all arise from a foreign and contradictory element inadvertently

admitted as part of the conception, and that, this element being eliminated, the conception is in no respect inconceivable, nor does it present any difficulties not presented by any other concept or general notion.

CHAPTER II.

THE CONCEPTION NOT QUANTITATIVE.

We will suppose two parallel straight lines, A and B, unlimited in extent, and intersected by perpendiculars, ab, $a'b'$, $a''b''$, etc., drawn at equal distances from each other.



It is evident that each division upon A is equal to its corresponding division upon B, and the sum of any number of divisions upon A will equal the sum of a similar number upon B. Since, therefore, each division upon the one line has its corresponding division upon the other, will not the equation hold good when all the divisions are considered? That is, will not the sum of all the divisions upon A be equal to the sum of all the divisions upon B? And will not the sum of all the divisions on both lines be equal to twice the sum of all the divisions on either? Must we not here regard one infinite as greater than another?

Much depends upon the answer to this question, as it will reveal very clearly the content which one attributes to his conception of the infinite. To the giving of the wrong answer may be traced that misconception of the true nature of infinity which has been such a fruitful source of supposed antinomies. Broadly stated, the question is, Can infinites be regarded as comparable with each other, as greater or less than, or equal to, each other? Let us consider the case of the parallel lines.

It is true that we must consider each division on the one line equal to each division on the other; and taking any number of divisions on the one, and adding them to an equal number of divisions on the other, we obtain a sum equal to twice the number of given divisions on either. But when we say "*all* the divisions on the one are equal to *all* the divisions on the other," we speak of the lines as quantitative wholes, and introduce an error with the word *all*.

To conceive of a thing *as a whole*, we must assign to it limits. In saying "the whole" of any object, we refer to those limits beyond which there is none of that object. In re-

garding any object as a quantitative whole, we necessarily think it as finite.

When we compare one line with another, and declare its extent greater or less than that of the other, we mean that, when the one is applied to the other, its limits extend beyond or fall within the limits of the other. In other words, we give the difference between the distances included within their respective limits. Measuring is merely giving the distance between limits. When two lines are infinite, we have no point to measure from, and no point to measure to, and no measurement—therefore no comparison—is possible. It is a palpable contradiction to compare (*i.e.*, give relations of measurement between the respective limits of) two infinites (*i.e.*, things which cannot be measured, as having no limits).

The terms longer, *shorter*, *and equal*, *can*, *therefore*, *have no meaning as applied to infinite lines.* They can be used only in speaking of the finite. We cannot, then, say that one infinite is greater or less than another, and just as little can we say that all infinites are equal; for any such proposition, however possible in words, is im-

possible in thought, and is an attempt to join contradictory notions.

In such cases as the above, where the lines are nowhere limited, the impossibility of an increase in length may be clearly seen. A line can only be lengthened by adding to it at its extremities, and it is impossible that a line without ends should be added to. If one holds that the sum of two such lines is greater than either line separately, he simply states that that may be increased, the very conception of which precludes the possibility of its increase.

There are cases, however, in which the error of a wrong conclusion is not so immediately palpable as in the case just stated; for example, the case of a line limited at but one point. Suppose the line AB limited only at the point A.

Continue the line to C. If now the line be divided by points placed at equal distances from each other, into equal divisions, AC containing three such divisions, will not the whole line CB be greater by three divisions than the whole line AB?

AB is limited at A; consequently there is

nothing to prevent our adding to it at its one extremity. Does it not seem natural to assume that in thus adding we increase the sum total of the line? We have gone through the same process as that by which we increase finite lines.

When we recollect, however, that the line AB is limited only at one point, and is not, therefore, as a line, defined (for two points are necessary to define a line), the impossibility of regarding it as a quantitative whole is evident, and the impossibility of increasing or diminishing its length, as a whole, necessarily follows. All of CB is not greater than all of AB, because the word *all* (in its quantitative sense*) cannot be applied to either.

Suppose we attempt a comparison of the two lines. Let CB be superposed upon AB in such a manner that C will fall upon A. The two lines will then have the one limit in common; but one limit does not furnish data for lineal comparison, and no judgment can be formed as to the comparative length of the two lines. Where the one line is regarded as

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* This distinction will be noticed later.

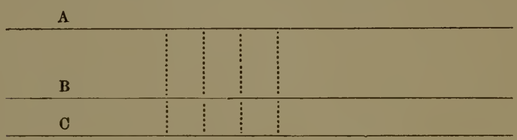
greater than the other, from the fact that three of its divisions project beyond the limit of the other, the measurement begins with an imagined point infinity, which is regarded as a common limit of the other extremities of the two lines, and concludes with the limits at A and C.

The error of such an attempt at measurement is clearly revealed by beginning to measure at A and C. It may be here justly remarked that we have before us a concrete instance of the truth of the old adage, that is a poor (measuring) rule that will not work both ways. The illusion disappears when we begin to measure at the given and only limits.

Now, drawing the necessary inference from the foregoing, we may answer the question whether a line altogether without limits is not greater than a line limited at but one point, by saying that the very nature of the conceptions precludes the possibility of the words *greater* or *less* being applied to either; that neither of the lines can be regarded as a quantitative whole; and that, consequently, the question is a meaningless one.

When we turn our attention to the considera-

tion of surfaces, we meet with similar misapprehensions, and arising from the same cause. It is asserted, for example, that if we suppose three parallel straight lines infinite in extent, one of which, A, is separated by a distance of two metres from the middle line B, while the other, C, is distant from it but one metre,

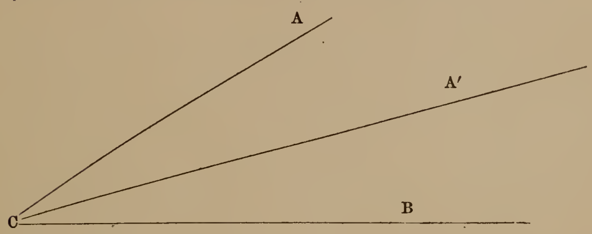


we must conclude that the surface included between A and B is double the surface included between B and C.

Let the lines be intersected by perpendiculars one metre apart. Do we not find that each metre in the surface contained between B and C has corresponding to it two metres in the other surface? And is not this proportion the same for twenty or for two hundred as for one, and quite independent of number? If, then, the proportion will hold good for any number, why will it not hold good when all the divisions are considered?

Or if we consider the angle formed by the

intersection of two infinite straight lines, must we not conclude that increasing the angle will increase the area of the surface included between the lines forming the angle? And that diminishing the angle will diminish the area of the included surface? Let A and B be two infinite straight lines intersecting at C;



and let A′C be an infinite straight line, making a smaller angle with CB than is made by AC. Must we not affirm that the surface ACB is greater than the surface A′CB, and that it is equal to the sum of A′CB and ACA′? The answer to be given in these cases is evidently the answer which has already been given. The quantitative relations of equality and inequality certainly hold good for all quantities; and taking the case of the surfaces included between the parallel lines, we must admit that any number of the divisions between A and B will

have double the area of a similar number of the divisions between B and C; but when we speak of *all* the divisions, we do not refer to any number; we do not express by the word a *quantity;* and where the notion of quantity is wanting, manifestly, quantitative relations will not hold. So in the case of the increased or diminished angle.

It is unnecessary to multiply instances, as the principle is in all cases the same. In general, wherever the limit is removed in any one direction, whether in the case of lines, of surfaces, or of solids, the object can no longer be regarded as a quantitative whole, and is not to be considered finite.

It remains to consider a class of cases of an apparently different nature. It is argued, for example, that an infinite series of dollars will exceed in value an infinite series of cents; that, where the unit differs, the difference will extend to the series in its totality. It is easy to show the error of such a position by showing what the assertion necessarily involves.

Suppose that, instead of counting one cent in the one series to each dollar in the other, we vary our mode of procedure by counting one

hundred cents in the one to each dollar in the other. It is true that the one series is exhausted one hundred times as rapidly as the other; but since they are both infinite (will never end), we may continue this forever (to infinity), and the two series will have equal values. Or we may count two hundred cents in the one to each dollar in the other, or three hundred, or four hundred; so that the same series will be equal to, or twice, thrice, or four times as great as another; its value depending merely on the mode of reckoning.

If it is just to conclude that an infinite series of dollars is one hundred times as great in value as an infinite series of cents, we must also accept the conclusions arrived at by the other modes of reckoning; all are based on the same principle. Our only escape from these warring conclusions is to declare the principle underlying all the modes of reckoning an erroneous one. The error lies in regarding these infinite series as in any way capable of being compared with each other; in looking upon them as quantitative wholes.

It is clear, therefore, that the true conception of the infinite is not *quantitative* but *qualitative*, a

fact which has been very generally overlooked, and with disastrous consequences much the same in all cases.

I have said that the word "whole," as predicating totality, cannot be applied to infinites, as naturally follows from the qualitative nature of our conception of the infinite; but the use of the word is really unavoidable, and, when it is used, it should be borne in mind that, applied to infinites, the word has a certain qualitative sense quite different from that in which it is applied to finites. If I were to speak of "all (possible) men" as distinguished from "all (possible) angels," I should in no respect limit the infinite possible number of either (the phrase "infinite number," though, strictly considered, incorrect, may be understood in a qualitative sense as expressing unlimited units), and my conceptions would be not quantitative but qualitative; for the mind would be occupied, not with the number of objects, but with certain conditions which any object must satisfy to fall within the one class or the other.

If, on the other hand, I speak of "all (actual) men," I may regard them as a definite known

or unknown number, and my conception may be quantitative.

Similarly, if, in speaking of an infinite line, I should say, "the line AB is *in all its parts* a straight line," I could only mean that one of the general conditions of the line is straightness, and that, whatever part of it may be thought of, it must agree with this condition. The "all" is in such a case equivalent to "any," not to "every," and the two meanings may easily be confounded. Indeed, there is something misleading in the very expression, "an infinite line," for the unreflective mind is apt to regard the object to which it is applied as a unit, a whole; it is very necessary in using the phrase to keep in mind its true meaning.

It may be objected to what I have here brought forward that any theory which denies that we have knowledge of the infinite as a whole may justly be called agnostic. If we do not know the infinite as a whole, do we not know only its parts, which are finite? And have we any true knowledge of the infinite at all?

I answer, the conception of a part is, as well as the conception of a whole, quantitative, and

an object recognized as part of a greater object is thereby necessarily recognized as finite. But if the object before the mind is not quantitatively regarded at all, either as whole or part, our conception may be of the infinite. The plausibility of the objection arises from its confounding two very different things, the distinction between which will be more clearly drawn in the last chapter of this monograph.

But as a preliminary answer to the objection, I may say that the assertion that we do not know the infinite as a whole is by no means equivalent to the assertion that we do not know the infinite. We do not know the moon as square, but that would scarcely prove that we have no knowledge of the moon, since the notion of squareness forms no part of a true knowledge of that object. Just as little is the quantitative conception of totality necessary to a knowledge of the infinite.

It is not agnosticism to declare the mind unable to think that which is in its nature self-contradictory,—to define an object as infinite and then think it as limited; while, on the other hand, any theory which maintains that we may know *as a whole* that which, in its very

conception, precludes the possibility of its being so considered, may be accused of the direst agnosticism, as discrediting a fundamental law of thought, the law of non-contradiction. The theory attacked may as a last resource avail itself of the old *argumentum ad hominem*, and remark in pointed terms that the kettle is not as black as some other vessels in the speculative kitchen.

CHAPTER III.

THE ANTINOMIES OF HAMILTON.

THE evils resulting from overlooking the fact that the conception of the infinite is qualitative are evident when we examine some of the reasonings based upon the supposition that the conception contains a quantitative element.

And first I will examine, so far as it touches the point in question, that agnostic theory developed by Sir William Hamilton under the name of the Philosophy of the Conditioned, the fundamental principle of which is that "all that is conceivable in thought lies between two extremes, which, as contradictory of each other, cannot both be true, but of which, as mutual contradictories, one must."* The reasoning in which Sir William applies this principle to our knowledge of space is worthy of attention. ". . . We conceive space," he says, —"we cannot but conceive space. I admit, therefore, that Space, indefinitely, is a positive

* "Metaphysics," New York, 1880, p. 527.

and necessary form of thought. But when philosophers convert the fact, that we cannot but think space, or, to express it differently, that we are unable to imagine anything out of space,—when philosophers, I say, convert this fact with the assertion that we have a notion, a positive notion, of absolute or of infinite space, they assume not only what is not contained in the phenomenon, nay, they assume what is the very reverse of what the phenomenon manifests. It is plain that space must either be bounded or not bounded. These are contradictory alternatives; on the principle of Contradiction they cannot both be true, and, on the principle of Excluded Middle, one must be true. This cannot be denied without denying the primary laws of intelligence. But though space must be admitted to be necessarily either finite or infinite, we are able to conceive the possibility neither of its finitude nor of its infinity.

"We are altogether unable to conceive space as bounded,—as finite; that is, as a whole, beyond which there is no further space. Every one is conscious that this is impossible. It contradicts also the supposition of space as a neces-

sary notion; for if we could imagine space as a terminated sphere, and that sphere not itself enclosed in a surrounding space, we should not be obliged to think everything in space; and, on the contrary, if we did imagine this terminated sphere as itself in space, in that case we should not have actually conceived all space as a bounded whole. The one contradictory is thus found inconceivable; we cannot conceive space as positively limited.

"On the other hand, we are equally powerless to realize in thought the possibility of the opposite contradictory; we cannot conceive space as infinite, as without limits. You may launch out in thought beyond the solar walk, you may transcend in fancy even the universe of matter, and rise from sphere to sphere in the region of empty space, until imagination sinks exhausted; with all this, what have you done? You have never gone beyond the finite; you have attained, at best, only to the indefinite, and the indefinite, however expanded, is still always the finite. . . . Now, then, both contradictories are equally inconceivable, and, could we limit our attention to one alone, we should deem it at once impossible and absurd, and suppose its unknown opposite

as necessarily true. But as we not only can, but are, constrained to consider both, we find that both are equally incomprehensible; and yet, though unable to view either as possible, we are forced by a higher law to admit that one, but one only, is necessary."

It is evident that the difficulties in which Sir William has here involved himself are gratuitous. The argument used to prove the latter of the contradictories inconceivable breaks down upon a careful examination.

We may indeed "rise from sphere to sphere in the region of empty space" without transcending the finite, and the attempt to thus transcend it is as hopeless as would be the attempt of the peacock to escape from his feet by flying; for we cannot arrive at the unlimited while we carry our limits with us. Each successive stage simply places the limits farther apart, and in no respect helps us to do away with them altogether.

Such a mode of procedure forcibly reminds one of the amusing person in Chamisso's poem, who supposed that by quickly turning himself around he could cause his queue to hang in front:

"Er dreht sich links, er dreht sich rechts,
Es thut nichts Gut's, es thut nichts Schlecht's,
Der Zopf, der hängt ihm hinten."

And how analogous is the condition of one who thinks that the way to reach the infinite is to endlessly continue this hopeless journey beyond "the universe of matter" to that of the hero as portrayed in the last verse:

"Und seht, er dreht sich immer noch
Und denkt: es hilft am Ende doch—
Der Zopf, der hängt ihm hinten."

It is not by adding space to space that we arrive at the idea of infinite space, and imagination may well sink exhausted in the attempt to find the end of the endless.

But to think space as infinite it is by no means necessary to take this journey; and, so far from proving that we cannot regard our notion of space as infinite, the failure of any such attempt to know it as a whole is the surest evidence that it is indeed infinite. The latter of the contradictories is thus found to be inconceivable only when we suppose a quantitative element in our conception of infinity, and, this error corrected, the antinomy disappears.

Sir William has applied the Law of the Conditioned also to the minimum of space: "That the conceivable," he continues, "lies always between two inconceivable extremes is illustrated by every other relation of thought. We have found the maximum of space incomprehensible; can we comprehend its minimum? This is equally impossible. Here, likewise, we recoil from one inconceivable contradictory only to infringe upon another. Let us take a portion of space, however small; we can never conceive it as the smallest. It is necessarily extended, and may, consequently, be divided into a half or quarters, and each of these halves or quarters may again be divided into other halves or quarters, and this *ad infinitum.* But if we are unable to construe to our mind the possibility of an absolute minimum of space, we can as little represent to ourselves the possibility of an infinite divisibility of any extended entity."

This is at bottom a repetition of the abovementioned error. Whether we regard space, with the Kantian, as in its nature wholly composite, and always capable of further subdivision; or, with the Berkeleyan, as composed of *minima visibilia*, themselves not admitting of subdi-

vision; in neither case are we forced into a choice of two inconceivables. The Berkeleyan would claim that the difficulty of conceiving a component part of any extended entity as itself non-extended arises from the fact that the *minimum visibile* is represented by the imagination as extended, the notion of extension being carried over from those objects to which it rightfully belongs, and is, consequently, not a true *minimum visibile.* While the Kantian may maintain that there is nothing inconceivable in an infinite series, rightly understood. If we suppose a series to be infinite, we cannot, of course, represent it to the mind as a completed whole, but that is unnecessary to—it is incompatible with—our recognition of it as infinite. The antinomy, like its predecessor, disappears as soon as it is recognized that there is no quantitative element in our conception of the infinite.*

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* It is odd that the statement that we are unable to conceive of any portion of space as the smallest possible, and not itself divisible into spaces, should be so constantly allowed to pass unchallenged. The question is not whether, when we have carried our subdivision of a surface so far that an apparently unextended point of color alone is left, we can in imagination substitute for that point an extended

Let us turn now to Sir William's application of the law to our conception of time:

"In like manner Time;—this is a notion even more universal than space, for while we exempt from occupying space the energies of mind, we are unable to conceive these as not occupying time. Thus, we think everything, mental and material, as in time, and out of time we can think nothing. But if we attempt to comprehend time, either in whole or in

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surface, and proceed to subdivide *that*, convinced that any system of relations developed from the latter may lawfully be carried over to all possible future experiences of a similar nature which may be found to be connected with the former; but the question really is whether we can conceive that identical apparently unextended spot of color divided and subdivided. If we say that we are dividing it in thought, when our division proceeds thus through a representative, if we insist in applying to the object in the two cases the word "same," we should never forget that we are using this highly ambiguous word "same" in by no means the strictest of its four distinctly different senses, and should be very sure that we are not practising self-deception by juggling with the several meanings of the word. When I say that with the Kantian hypothesis we are not forced to embrace what is inconceivable, I refer only to the point under consideration, the conceivability of an endless series *per se*.

part, we find that thought is hedged in between two incomprehensibles. Let us try the whole. And here let us look back,—let us consider time *a parte ante.* And here we may surely flatter ourselves that we shall be able to conceive time as a whole, for here we have the past period bounded by the present; the past cannot, therefore, be infinite or eternal, for a bounded infinite is a contradiction. But we shall deceive ourselves. We are altogether unable to conceive time as commencing; we can easily represent to ourselves time under any relative limitation of commencement and termination, but we are conscious to ourselves of nothing more clearly, than that it would be equally possible to think without thought, as to construe to the mind an absolute commencement or an absolute termination of time,—that is, a beginning and an end beyond which time is conceived as non-existent. Goad imagination to the utmost, it still sinks paralyzed within the bounds of time, and time survives as the condition of the thought itself in which we annihilate the universe. On the other hand, the concept of past time as without limit—without commencement—is equally impossible. We

cannot conceive the infinite regress of time; for such a notion could only be realized by the infinite addition in thought of finite times, and such an addition would itself require an eternity for its accomplishment. If we dream of effecting this, we only deceive ourselves by substituting the indefinite for the infinite, than which no two notions can be more opposed.

"The negation of a commencement of time involves, likewise, the affirmation that an infinite time has, at every moment, already run; that is, it implies the contradiction that an infinite has been completed. For the same reasons we are unable to conceive an infinite progress of time; while the infinite regress and the infinite progress, taken together, involve the triple contradiction of an infinite concluded, of an infinite commencing, and of two infinities not exclusive of each other." *

The statement that past time cannot be regarded as infinite because limited by the present

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* "Metaphysics," New York, 1880, p. 529. See, also, Spencer's "First Principles," New York, 1875, p. 81: "How self-destructive is the assumption of two or more Infinites is manifest on remembering that such Infinites, by limiting each other, would become finite."

is based upon the erroneous supposition that what is limited at one point cannot be infinite. But, as has been shown in the preceding chapter, one point is not sufficient to define a line as finite; and time, which we represent to ourselves under the form of a continuous line, must be regarded as infinite unless limited at two points. Time past and time future are two infinites, and as such are perfectly conceivable. The difficulty respecting the possibility of two infinites mutually exclusive of each other is a difficulty only under a false conception of the infinites as quantitative wholes, and may easily be made to disappear.

The assertion, also, that the past cannot be infinite, as "a bounded infinite is a contradiction," may well be scanned. Arguments drawn from the etymological signification of a word are of small value, unless that expresses the true and whole content of the word. That such is not the case here is evident. A line limited at but one point is certainly not finite, for it cannot be regarded as a quantitative whole; cannot be increased, diminished, or compared in length with other lines; in short, is not subject to the conditions of the finite. If, then, for

etymological reasons, we exclude it from the class of infinites, we have the finite, the infinite, and a third class, a *tertium quid*, which lies between the two, and might be humorously described as "infinite at one end." But, etymology aside, there is no difficulty in classing such a line with one that has no limits, for they are subject to precisely the same conditions, and equally distinct from the finite. However, the appellation is a matter of taste; the thing which it is important to bear in mind is that a line with but one limit can no more be regarded as a quantitative whole than a line absolutely without limits; and, whether we choose to call time past infinite or finite, that we may have a clear knowledge of it as limited only by the present, without attempting to pass over its parts in succession, and thus arrive at the whole.

The assertion, too, that the negation of a commencement of time "implies the contradiction that an infinite has been completed" is misleading. The word "completed" is an unfortunate one to use in this connection, as it suggests to the mind the idea of progression from a beginning to an end. The denial of a commencement of time does imply that an in-

finite is *past*, but not that it is completed in any such sense as to be enclosed within limits, for it is quite conceivable that it never began, and the present moment is, by supposition, the only limit.

Removing the misconceptions just noticed, the whole force of the argument for the inconceivability of time as infinite lies, as in the former cases, in the idea that the infinite may only be known by exhausting it in its totality, through the successive addition of its finite parts, and this antinomy also proves to be a gratuitous one.

Having argued thus far the inconceivability of time as a *maximum*, Sir William turns, as in his discussion on our knowledge of space, to a consideration of time as a *minimum:*

"Now take the parts of time,—a moment, for instance; this we must conceive as either divisible to infinity, or that it is made up of certain absolutely smallest parts. One or other of these contradictories must be the case. But each is, to us, equally inconceivable. Time is a protensive quantity, and, consequently, any part of it, however small, cannot, without a contradiction, be imagined as not divisible into parts, and

these parts into others *ad infinitum.* But the opposite alternative is equally impossible; we cannot think this infinite division. One is necessarily true; but neither can be conceived possible. It is on the inability of the mind to conceive either the ultimate indivisibility or the endless divisibility of space and time that the arguments of the Eleatic Zeno against the possibility of motion are founded, — arguments which at least show that motion, however certain as a fact, cannot be conceived possible, as it involves a contradiction." *

With reference to the former of the alternatives offered, the Berkeleyan would answer that in the case of "protensive" quantity, as in the case of extensive, the difficulty of conceiving the unit, itself indivisible, lies in the imagination, and may, with precautions, be obviated; while with reference to the latter, the Kantian may answer, as before, that an infinite series is not intrinsically unthinkable. I may remark, *en passant*, that the idea of the theoretic impossibility of motion is a wholly erroneous one, and falls with the errors upon which it is based.

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* "Metaphysics," New York, 1880, pp. 529–30.

One turns from an examination of Sir William Hamilton's application of the Law of the Conditioned to Space and Time with a conviction that if the Philosophy of the Conditioned has no better props to sustain it than these prove to be, it may turn out as insecure an edifice as the house that a certain foolish individual, according to the parable, founded upon sand.

It remains to consider a case which apparently militates against the theory that an infinite series can never be regarded as a whole.

Let us suppose a point moving uniformly along the line AB, over the whole of which it can pass in one minute. In $\frac{1}{2}$ of a minute it will have passed over $\frac{1}{2}$ of the line; in $\frac{1}{4}$ of a minute over $\frac{1}{4}$ more; in $\frac{1}{8}$ of a minute $\frac{1}{8}$ more, etc. When the minute is completed the point will have passed over the whole line. Has it not passed successively over the whole series, thus completing it, and arriving at 0 as its lower limit? And may we not say that the sum of all the terms in the series is equal to the whole line passed over? Is it not a quantitative whole?

A little reflection will reveal the fallacy in this reasoning. The series is not completed at all, but is truly infinite. It is limited at one point by the highest term ($\frac{1}{2}$), but is not limited at another point by a lowest term (0); for the 0 can only be assumed as a limit to the series by breaking the law of the series, which is that each term shall be half as great as the one preceding.

We can never, by halving something, arrive at nothing; a division of substance will never give us that which is not substance. The 0, since it does not make one in the series, cannot limit the series. The Kantian may answer the question by reasoning as follows: "The point in question passes over the whole line, not by completing the descending series until it arrives at a lowest term in the simple, and from that passes to zero, but by the successive addition of spaces, which are themselves composites. 'As space is not a composite of substances (and not even of real accidents), if I abstract all composition therein, nothing, not even a point, remains; for a point is possible only as the limit of a space, consequently of a composite. Space and time, therefore, do not consist of simple

parts.'* We cannot, therefore, consider any member of the series in question as the smallest possible, nor the zero as a limit to the series; nor can we regard the series as in any sense completed. If we discontinue the subdivision at any point whatever, we may justly say that the foot or yard contains all the terms of the series. But when the point has reached zero it has reached it by breaking the series, not completing it. A completion the law of the series renders impossible."† And the Berkeleyan might answer that a completion of the journey along the line by no means implies the com-

* "Critique of Pure Reason;" "Observations on the Second Antinomy." Ed. Bohn.

† How the series can be broken by the progressive motion of a point over a line when, by hypothesis, the series is throughout applicable to the line (*i.e.*, the line is infinitely divisible), the Kantian must explain; the fact remains that neither an infinite nor a finite series can be completed by breaking its law; and unless we claim that half of something, when that something is small enough, is equal to nothing, we cannot bring the zero into the series as a limit. Even could this be done, the desired point would not be established, as the series would then be limited at two points, consequently in no sense infinite, and its completion would not be the completion of an infinite series.

pletion of an infinite series, since relations of quantity may be divorced from all content of actual being, and used purely symbolically. Mathematical reasonings, he would argue, are applicable to space only within the limits of a possible perception; and the series may be truly infinite, though the line-portions to which some of its members are applied may be limited in number.*

This problem, it will be seen, is simply the old puzzle of Achilles and the tortoise in a somewhat altered dress; and the answer which has been given to that puzzle—that the series "runs into infinitesimals which are practical zeros, and, even if theoretically infinite in number, really are all included in that finite length which Achilles will quickly get over"†—is not

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* I have taken up both positions to bring out clearly the fact that the assertion of the conceivability of the infinite is quite independent of metaphysical theories as to the nature of space and time. The two questions are wholly distinct, as I will show in a later chapter.

† This is the solution of the problem given by the late Professor Atwater in his little book on the Elements of Logic. The explanation offered could not be much worse. What, we may ask, is meant by a "practical zero," as distinguished from a theoretical? And the assertion that these practical

a true answer, since it regards a series, which, by the very terms of its statement, is incapable of completion, as a completed whole and equal to a finite.

Before leaving this chapter I would remark that it is interesting to notice how wide-spread has been the conviction that the only way to arrive at a cognition of the infinite is to proceed to an endless addition of finites. Hamilton, we have seen, makes much of it. Mansel follows in his footsteps. Mr. Herbert Spencer quotes him with approbation. And Kant, as we shall see in the chapter following, did not escape the snare. "To have actually in the mind the idea of a space infinite," says Locke,* "is to suppose the mind already passed over, and actually to have a view of all those repeated ideas of space which an endless repetition can never totally represent to it;—which carries in it a plain contradiction."

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zeros, "even if theoretically infinite in number, really are all included" in a finite length, would seem to draw a distinction between the theoretical and the practical, little to the advantage of the former. It makes *theoretical* about equivalent to *unreal.*

* "Essay concerning Human Understanding," book ii. ch. xvii., § 7.

CHAPTER IV.

KANT, MILL, AND CLIFFORD.

It is interesting to notice that the truth that our conception of infinity contains no quantitative element has been seen, like Thule, "through the mist" by several acute minds, who have yet not seen the truth with sufficient clearness to escape the common errors arising from the introduction of the contradictory element into their discussions. Immanuel Kant, although he has based the proof of the thesis of his first antinomy on a false conception of infinity, and although, after correctly criticising the false conception, he himself lapses into it, yet perceived, and in so many words gave expression to the fact, that the conception of the infinite is qualitative.

The thesis of the first antinomy maintains that the world has a beginning in time, and is limited with regard to space, both of which propositions are denied in the antithesis. The proofs offered in support of the antithesis may

be passed over as extraneous to the subject; those in support of the thesis I will quote, not for the purpose of again refuting them, for they are identical with those used by Sir William Hamilton in his antinomies, but that I may give the observations appended to them, which are very significant in their contextual connection. The proof proceeds by assuming the truth of the antithesis, and then showing it to be impossible.

"Granted, that the world has no beginning in time; up to every given moment of time an eternity must have elapsed, and therewith passed away an infinite series of successive conditions or states of things in the world. Now, the infinity of a series consists in the fact that it never can be completed by means of a successive synthesis. It follows that an infinite series, already elapsed, is impossible, and that consequently a beginning of the world is a necessary condition of its existence. And this was the first thing to be proved.

"As regards the second, let us take the opposite for granted. In this case the world must be an infinite given total of coexistent things.

Now, we cannot cogitate the dimensions of a quantity, which is not given within certain limits of an intuition, in any other way than by means of the synthesis of its parts,* and the total of such a quantity only by means of a completed synthesis or the repeated addition of unity to itself. Accordingly, to cogitate the world, which fills all spaces, as a whole, the successive synthesis of the parts of an infinite world must be looked upon as completed,—that is to say, an infinite time must be regarded as having elapsed in the enumeration of all co-existing things, which is impossible. For this reason an infinite aggregate of actual things cannot be considered as a given whole, consequently, not as a contemporaneously given whole. The world is consequently, as regards extension in space, *not infinite*, but enclosed in

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* Kant says, in a foot-note, "We may consider an undetermined quantity as a whole when it is enclosed within limits, although we cannot construct or ascertain its totality by measurement,—that is, by the successive synthesis of its parts. For its limits of themselves determine its completeness as a whole." This method being absent in the case of infinites, Kant thinks he can cognize them only by falling back upon the successive synthesis.

limits. And this was the second thing to be proved."*

It will be noticed that in the first part of the proof the word *completed* (vollendet) is used in the manner before objected to as misleading. When we speak of a series as "completed by means of a successive synthesis," we are apt to regard it as a whole, with a beginning as well as an end. Of course, when we are considering time past as limited by the present, such an association must be unfortunate.

The observations on the thesis are the following:

"In bringing forward these conflicting arguments, I have not been on the search for sophisms for the purpose of availing myself of special pleading, which takes advantage of the carelessness of the opposite party, appeals to a misunderstood statute, and erects its unrighteous claims upon an unfair interpretation. Both proofs originate fairly from the nature of the case, and the advantage presented by the mis-

* Immanuel Kant's "Sâmmtliche Werke," Leipzig, 1867. Dritter Band, s. 304. I have taken the rendering of Meiklejohn's excellent translation of the Critique.

takes of the dogmatists of both parties has been completely set aside.

"The thesis might also have been unfairly demonstrated by the introduction of an erroneous conception of the infinity of a given quantity. A quantity is infinite if a greater than itself cannot possibly exist. The quantity is measured by the number of given units—which are taken as a standard—contained in it. Now, no number can be the greatest, because one or more units can always be added. It follows that an infinite given quantity, consequently an infinite world (both as regards time and extension), is impossible. It is, therefore, limited in both respects. In this manner I might have conducted my proof; but the conception given in it does not agree with the true conception of an infinite whole. In this there is no representation of its quantity; it is not said how large it is; consequently its conception is not the conception of a *maximum*. We cogitate in it merely its relation to an arbitrarily assumed unit, in relation to which it is greater than any number. Now, just as the unit which is taken is greater or smaller, the infinite will be greater or smaller; but the in-

finity, which consists merely in the relation to this given unit, must remain always the same, although the absolute quantity of the whole is not thereby cognized.

"The true (transcendental) conception of infinity is that the successive synthesis of unity in the measurement of a given quantum can never be completed. Hence it follows, without possibility of mistake, that an eternity of actual successive states up to a given (the present) moment cannot have elapsed, and that the world must therefore have a beginning.

"In regard to the second part of the thesis, the difficulty as to an infinite and yet elapsed series disappears; for the manifold of a world infinite in extension is contemporaneously given. But, in order to cogitate the total of this manifold, as we cannot have the aid of limits constituting by themselves this total in intuition, we are obliged to give some account of our conception, which in this case cannot proceed from the whole to the determined quantity of the parts, but must demonstrate the possibility of a whole by means of a successive synthesis of the parts. But as this synthesis must constitute a series that cannot be completed, it is impossible

for us to cogitate prior to it, and consequently not by means of it, a totality. For the conception of totality itself is in the present case the representation of a completed synthesis of the parts; and this completion, and consequently its conception, is impossible."

We here find brought forward a conception of the infinite which is declared faulty; a declaration of the point in which it differs from the true conception; and a statement of what, according to Kant, the true conception really is. The false conception is that "a quantity is infinite if a greater than itself cannot possibly exist." We may readily see that such a conception gives us, not an infinite, but a finite. Not only is the word "greater" inapplicable to infinites, but the very expression, "a *quantity* is infinite," is absurd, as involving a flat contradiction. Kant was too keen a thinker not to see that that which admits of an addition ot units, and, consequently, of increase as a whole, cannot be infinite. This does not agree, he says, with the true conception of the infinite, in which "there is no representation of its quantity; it is not said how large it is; consequently its conception is not the conception of a *maximum.*"

Could there be a clearer recognition of the fact that the conception is not quantitative?

But it is evident that Kant did not see the full force and the logical consequences of the statement. In the sentence immediately preceding the one in which he recognizes the qualitative character of the conception he uses the phrase "an infinite whole;" and in the sentences immediately following he brings forward a conception faulty in precisely the same respect as the one criticised: "We cogitate in it merely its relation to an arbitrarily assumed unit, in relation to which it is greater than any number. Now, just as the unit which is taken is greater or smaller, the infinite will be greater or smaller; but the infinity, which consists merely in the relation to this given unit, must remain always the same, although the absolute quantity of the whole is not thereby cognized." That is, if we designate the infinite by a, the unit by b, and the infinity (the relation between a and b) by x, we find that a varies as b, and x remains always the same (this can only mean numerically the same). In this case x is simply an indefinite number, and the "absolute quantity of the whole" can certainly be cognized. When we say the infinity remains

always the same, the question naturally arises, The same in what? In amount? If so, we have but the finite. The error is here, perhaps, not quite so palpable, but is just as real as in the case which Kant criticises, and it is of precisely the same nature. Both parts of the proof given in support of the thesis of course fall to the ground when this error is rectified.

The last two observations are merely a restatement of the proofs of the thesis. The remark made in the last one, that "in order to cogitate the total of this manifold, as we cannot have the aid of limits constituting by themselves this total in intuition, we are obliged to give some account of our conception, which in this case cannot proceed from the whole to the determined quantity of the parts, but must demonstrate the possibility of a whole by means of a successive synthesis of the parts," will lose all its force if the words "total" and "whole" are abstracted. If this manifold is to be known as a whole, we cannot, of course, arrive at a knowledge of it without cogitating all its parts as contained in it; but since it is impossible that it should be *thus* known, we may "give some account of our conception" by simply

stating that it is without limit, thus recognizing it as truly infinite.

It seems odd that Kant should have seen an error, objected to it, and fallen into it on the same page; should have said that there is no representation of its quantity in the conception of the infinite, and then have called it a whole; declared that the true conception does not say how large it is, and yet have affirmed that the infinity is always the same, while the series as a whole varies with the unit. But, with and in spite of all his inconsistencies, it must be allowed that he recognized the truth, however imperfectly, that our conception of the infinite has, properly speaking, no quantitative element, but is purely qualitative.

This truth has also been recognized, though less clearly, by John Stuart Mill, in his "Examination of Sir William Hamilton's Philosophy," where he maintains, in opposition to Sir William, that the infinite is not inconceivable; and we can see in the paragraphs which he devotes to the subject that Mill had in mind two distinct conceptions, a quantitative and a qualitative. To the former, which he calls the *adequate* conception, he acknowledges we cannot

attain; the latter, which is truly qualitative, though he does not apply to it that name, he claims to be a real conception, implying nothing inconceivable.

His attempted refutation of Hamilton's argument for the inconceivability of infinite space proceeds as follows:*

"Our author goes on to repeat the argument used in his reply to Cousin, that Infinite Space is inconceivable, because all the conception we are able to form of it is negative, and a negative conception is the same as no conception. 'The infinite is conceived only by the thinking away of every character by which the finite was conceived.' To this assertion I oppose my former reply. Instead of thinking away every character of the finite, we think away only the idea of an end, or a boundary. Sir W. Hamilton's proposition is true of 'The Infinite,' the meaningless abstraction, but it is not true of Infinite Space. In trying to form a conception of that, we do not think away its positive characters. We leave to it the character of space; all that

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* "Examination of Sir William Hamilton's Philosophy," Boston, 1868, vol. i., pp. 104 *et seq.*

belongs to it as space; its three dimensions, with all their geometrical properties. We leave to it also a character which belongs to it as Infinite, that of being greater than any other space. If an object which has these well-marked positive attributes is unthinkable, because it has a negative attribute as well, the number of thinkable objects must be remarkably small. Nearly all our positive conceptions which are at all complex include negative attributes. I do not mean merely the negatives which are implied in affirmatives, as in saying that snow is white we imply that it is not black, but independent negative attributes superadded to these, and which are so real that they are often the essential characters, or differentiæ, of classes. Our conception of dumb is of something which *cannot* speak; of the brutes, as of creatures which *have not* reason; of the mineral kingdom, as the part of nature which *has not* organization and life; of immortal, as that which *never* dies. Are all these examples of the Inconceivable? So false is it that to think a thing under a negation is to think it as unthinkable.

"In other passages, Sir W. Hamilton argues that we cannot conceive infinite space, because

we would require infinite time to do it in. It would of course require infinite time to carry our thoughts in succession over every part of infinite space. But on how many of our finite conceptions do we think it necessary to perform such an operation? Let us try the doctrine upon a complex whole, short of infinite, such as the number 695,788. Sir W. Hamilton would not, I suppose, have maintained that this number is inconceivable. How long did he think it would take to go over every separate unit of this whole, so as to obtain a perfect knowledge of that exact sum, as different from all other sums, either greater or less? Would he have said that we could have no conception of the sum until this process had been gone through? We could not indeed have an *adequate* conception. Accordingly, we never have an adequate conception of any real thing. But we have a *real* conception of an object if we conceive it by any of its attributes that are sufficient to distinguish it from all other things. We have a conception of any large number when we have conceived it by some one of its modes of composition, such as that indicated by the position of its digits. We seldom get nearer than this

to an adequate conception of any large number. But for all intellectual purposes this limited conception is sufficient; for it not only enables us to avoid confounding the number in our calculations with any other numerical whole,—even with those so nearly equal to it that no difference between them would be perceptible by sight or touch, unless the units were drawn up in a manner expressly adapted for displaying it,—but we can also, by means of this attribute of the number, ascertain and add to our conception as many more of its properties as we please. If, then, we can obtain a real conception of a finite whole without going through all its component parts, why deny us a real conception of an infinite whole because to go through them all is impossible? Not to mention that even in the case of the finite number, though the units composing it are limited, yet, Number being infinite, the possible modes of deriving any given number from other numbers are numerically infinite; and as all these are necessary parts of an adequate conception of any number, to render our conception even of this finite whole perfectly adequate would also require an infinite time.

" But though our conception of infinite space

can never be adequate, since we can never exhaust its parts, the conception, as far as it goes, is a real conception. We completely realize in imagination the various attributes composing it. We realize it as Space. We realize it as greater than any given space. We even realize it as endless, in an intelligible manner,—that is, we clearly represent to ourselves that however much of space has been already explored, and however much more of it we may imagine ourselves to traverse, we are no nearer to the end of it than we were at first time; however often we repeat the process of imagining distance extending in any direction from us, that process is always susceptible of being carried farther. This conception is both real and perfectly definite. It is not vague and indeterminate as a merely negative notion is. We possess it as completely as we possess any of our clearest conceptions, and we can avail ourselves of it as well for ulterior mental operations. As regards the Extent of Space, therefore, Sir W. Hamilton does not seem to have made out his point; one of the two contradictory hypotheses is not inconceivable."

One can see from these extracts how the

idea of quantity entered into and vitiated Mill's reasonings on the infinite. In arguing that our notion of infinite space is not a purely negative one, he enumerates several positive attributes that we leave to the conception when we take away the notion of limits. Among them he places "a character which belongs to it as Infinite, that of being greater than any other space." Evidently the character of being greater than any other space is a quantitative attribute, and can belong only to "a space,"—a finite; while space infinite is not, properly speaking, "a space" at all. Since it has no size, it cannot be marked out from other spaces by its size. And Mill has manifestly committed the same error which has misled Sir William Hamilton, whom he criticises, when he says that it would, of course, require infinite time to carry our thoughts in succession over every part of infinite space. How can we speak of *every part* of that which is not quantitatively considered? of that which, by its very definition, is incapable of being a whole? This is precisely Hamilton's error, and the cause of all his difficulties. A little farther on Mill speaks, just as Kant does, of "an infinite whole," never noticing the contradiction in

the adjective; and in the paragraph following he repeats a former blunder in the statement that our knowledge of space can never be adequate, "since we can never exhaust its parts."

It is strange that side by side with this conception of a quantitative infinite, which, to be adequately known, must be known as a whole, we should find a real, though imperfect, analysis of the true conception, and an affirmation of its conceivability. We can conceive infinite space, says Mill; we can conceive it as space, as greater than any given space, and even as endless, in an intelligible manner. When he comes to describe this intelligible manner of knowing the infinity of space, he uses an unfortunate phrase,—"we clearly represent to ourselves that however much of space has been already explored, . . . we are no nearer to the end of it than we were at first time;" dragging in the idea of a limit, which is also done, as above remarked, by calling it greater than any other space. But with all these concessions to the old erroneous doctrine, it cannot be denied that Mill held that we may know Infinite Space in some other way than by a successive synthesis of finite spaces, and that he

attempted an enumeration of the psychical elements comprehended by the conception, leaving out the notion of quantity. This conception, which he calls inadequate, but which he yet insists upon as a conception of the infinite, is qualitative, and harmonizes with the true character of our conception of infinity.

The last writer to whom I will advert as having had a knowledge, more or less clear, of the true qualitative nature of our conception of infinity, is Professor William Kingdon Clifford, by whose untimely death England has lost one of her acutest and most analytic minds. There are in his short paper entitled "Of Boundaries in General," and devoted to the point, the line, and the surface, some interesting and significant passages, which I will quote, and upon which I will afterwards comment. The passages are these:

"*Infinite;* it is a dreadful word, I know, until you find out that you are familiar with the thing which it expresses. In this place it means that between any two positions there is some intermediate position; between that and either of the others, again, there is some other intermediate; and so on *without any end.*

Infinite means without any end. If you went on with that work of counting forever, you would never get any farther than the beginning of it. At last you would have two positions very close together, but not the same; and the whole process might be gone over again, beginning with those as many times as you like." . . .

"In fact, when we said that there is an infinite number of points in a piece of line-room, we might have said a great deal more. Suppose, for instance, that any one said, 'How many miles is it possible to go up into space?' The answer would of course be, 'An infinite number of miles.' (Don't be frightened at this continual occurrence of the word infinite: it still means 'without any end,' and nothing more.) In this case, if you go a mile and count one, then another and count two, and so on, all we mean is that the process would never end. There would still be space left to go up into, however many millions of miles you had counted. But still all those miles would be counted and done with. Your task would have been distinctly begun, and there would be nothing more to say to the miles behind you.

But try now to count the points in a piece of line. You count one, two, three, four, a million points; and your task is not even begun. The line is all there, exactly as it was before; absolutely none of it is done with. The million points take up no more line-room than one point; that is to say, absolutely none at all. When, then, we are talking of the points in a piece of line, we must say not merely that there is a never-ending number of them (which there is), but that they are out of the reach of number altogether. All the points in a line are not, properly speaking, a number of points at all. If we are going to speak about the *number* of points in a line, we must settle beforehand that we are going to use the word in a new sense, which is not derived from counting, but from this very observation to which we have applied it.

"Let us now make use of our idea of a path. When a point moves along a line, we know that between any two positions of it there is an infinite number (in this new sense) of intermediate positions. That is because the motion is continuous. Each of those positions is where the point was at some instant or other.

Between the two end positions on the line, the point where the motion began and the point where it stopped, there is no point of the line which does not belong to that series. We have thus an infinite series of successive positions of a continuously moving point, and in that series are included all the points of a certain piece of line-room. May we say, then, that the line is made up of that infinite series of points?

"Yes, if we mean no more than that the series makes up the *points* of the line. But *no*, if we mean that the line is made up of those points in the same way that it is made up of a great many very small pieces of line. A point is not to be regarded as a *part* of a line in any sense whatever."

Evidently Clifford saw more clearly than either Kant or Mill that the notion of quantity is foreign to our conception of infinity. In speaking of the infinite extent of space, he does not make the absurd and tautological assertion that it would take an infinite time to exhaust it. He claims that when we call a thing infinite we know very well what it means: it means that it will never end. He never hints at any possibility of knowing the infinite as a whole, recog-

nizing very clearly that it cannot be a whole. And he has used the word number in two senses (which a careful reading of the extracts quoted will show to be a quantitative and a qualitative sense), to mark the difference between numbers regarded as constituent parts of wholes or sums total, and number regarded as unlimited units, having no relation to wholes of any sort,—a distinction which precisely corresponds to the distinction which I have drawn in a preceding chapter between the quantitative and the qualitative uses of the word *all*, as applied to the members of a finite and an infinite series.

But it is also evident that Clifford did not see the full significance and value of the truth which he recognized. This is clear from his contrasting the relation of linear miles to infinite extension with the relation of mathematical points to a line, and indicating that in the latter case there is something peculiarly hopeless in the attempt to complete the line by the addition of such points. "You count one, two, three, four, a million points, and your task is not even begun. The line is all there, exactly as it was before; absolutely none of it is done with." So that an addition of points, however long con-

tinued, *has no tendency* to exhaust and complete the line. But in the case of infinite space, says Clifford, if you go a mile and count one, another and count two, and so on, all those miles would be counted and done with. "Your task would have been distinctly begun, and there would be nothing more to say to the miles behind you." But in reality, so far as they touch the question under discussion, the two cases are precisely similar. The points have not even begun to exhaust the line, because their addition *has no tendency* to exhaust it; their number has nothing to do with it.* And, similarly, one may justly hold that the miles counted have not even begun to exhaust the infinite space, since the increasing of the distance between limits *has no tendency* to make them approach the limits of that which is without limit; since the adding of quantities *has no tendency* to make a sum equal to that which is not a quantity, and cannot be equal to anything; and since adding mile to mile *has no tendency* to successively exhaust the parts of that which is not a whole, and can have

* I speak here, of course, from Clifford's stand-point, assuming the infinite divisibility of a line.

no parts. The relation between a mile or a thousand miles and infinite extension is no closer than that which Clifford conceived between a point or a thousand points and a given line.

One cannot but see, therefore, that Clifford has not grasped the true nature of our conception of infinity in all its consequences so thoroughly as he might have grasped it. It will be seen, however, that he has discussed it more satisfactorily than either of the writers before cited.

CHAPTER V.

THE CONCEIVABLE AND THE EXISTENT.

There are two quite distinct questions to be considered in a discussion of the infinite: the first is, "Can an infinite object be conceived?" and the second, "Does any infinite object really exist?"

Now, one may answer the former of these questions in the affirmative, and yet not be committed to a similar answer to the latter. The two propositions, "An infinite object is conceivable" and "An infinite object exists" are different in nature, and when the former is affirmed the latter still calls for proof. One may hold that space is subjective, a mere abstraction from experience of extended objects, and that, consequently, space, together with the imaginary line in space, exists only as it is produced in thought; and, though he may on that account deny that that line, as an imagined object, or any actually existent line is infinite, or that any line could possibly be made infinite, he may yet claim that he can conceive an infinite line.

If he belong to one school of thought he will not only claim that he can conceive space infinite, but will assume on *à priori* grounds the infinity of space as actually existent. If he be an adherent of another school he may hold that the proposition "space is infinite" is incapable of proof, and that it can never be maintained; but he will not on that account deny that he can conceive infinite space. One may maintain that our assent to the former proposition is conditioned on our assent to the latter; that if the infinite be so unattainable and even contradictory a conception as Sir William Hamilton has held, we would have no reason to believe the existence of any infinite object either possible or actual; but certainly no one will hold that the first of the two propositions is so based upon the second as to necessarily stand or fall with it. The fact that I can imagine

> ". . . the Cannibals, that each other eat,
> The Anthropophagi, and men whose heads
> Do grow beneath their shoulders,"

does not prove such objects to have real existence. If men were only able to represent in the imagination what has its actual prototype in nature, how would we account for

"... the pert Fairies and the dapper Elves,"

and the cloud of unreal creatures with which the teeming poetic imagination of mankind has peopled the world from earliest ages? Where would be the dragon, the basilisk, the roc? Where the valley of diamonds and the palace of Aladdin? The fact that I can conceive such does not prove that in the whole realm of nature such objects may be found. And, similarly, the fact that no objects of a certain kind have an actual existence is no proof that the various qualities representing them may not be found grouped in mind, as a result of the constructive imagination. I may examine in detail all actual horses and all actual men, and conclude that the qualities belonging to each animal are distinct and separate; but I cannot assure myself that some one will not form in his mind the notion of an animal which combines the two sets of qualities,—a centaur. All our ideas are not derived in their ultimate shapes from direct intuition of objects, but are a result of mental processes, which recast and variously combine the material furnished. Though we may not be able to add a new element to those simple factors of our experience primarily given, yet it cannot be denied that

we may so transpose or combine, add or eliminate, some of those elements, as to obtain a product consisting of elements which are not found thus combined in any single object as it is presented to us in nature. When it is asked, therefore, whether any given conception be conceivable, it is not necessary to an affirmative answer that any existent object be proved to correspond to the conception. It is enough if the qualities connotated by the name be shown to be such as may be mentally put together, and are not mutually repugnant or contradictory. We find thus that the answer to the first question put forward at the beginning of the chapter does not depend on the answer to the second, and may be true though that be false. When it is asked whether the answer to the second question may not be independent of the answer to the first, the answer is not so clearly affirmative; and although this point is not directly connected with the subject under discussion, I may say, *à propos* of the antagonism which Hamilton exhibits between Faith and Knowledge, that, were the things of which I have just spoken so defined that I could in no conceivable manner put together in mind the contradictory qualities attributed to them, I

should be doing rashly to assume their existence, however veracious the witness upon whose word their acceptance might depend. It may be a question whether any force can be brought to bear upon a man sufficiently strong to lead him to believe in the actual existence of an inconceivable object; that is, of that which cannot become an object of thought, nor, consequently, of belief. However, the point in which we are now chiefly concerned is this: that there are a vast number of things that we have no reason to believe in as actually existent, but which we nevertheless conceive with perfect clearness.

Again. When we put forward the proposition that an infinite object is conceivable; unless we examine carefully this term used,—conceivable,—and find out whether it has one meaning or two, we may involve ourselves in serious error. If the word has two meanings, or is in common use made to stand for two distinct mental operations; when we prove that an infinite object is inconceivable in one acceptation of the term, we do not necessarily prove that it is inconceivable in the other sense. If we suppose that in its first meaning

the word conceivable is synonymous with imaginable, and that, when we eliminate one or more of the qualities which must be present to make an object imaginable, we may still in some way mentally grasp the remaining qualities, this second operation we may call conceiving in the second and narrower sense of the term. The proof that there is such a mental operation I will reserve for the next chapter.

Let it be marked, therefore, that we have to consider three distinct propositions, which demand as many distinct kinds of proof, when we assert, in the first place, that an infinite object is conceivable, and in the second, that an infinite object exists.

If I assert, then, that I can conceive space to be infinite, the proof should consist in an analysis of the qualities represented by the word, and an examination of their mutual consistency or repugnancy. If one element comprehended in the conception that I have in mind be the notion of a certain quantity, then I must in some way mentally grasp that quantity and connect it with the other elements in the conception, as I do in the representation in imagination of finite objects. If there be in

the conception no quantitative element, all that is necessary is the ability to grasp in thought certain qualitative elements, and this would fall under the second and narrower sense of the term conception. In either case the proposition depends on no fact of actual being, and may be established without any reference to such.

But the proposition "space is infinite" demands in the way of proof what the previous propositions can dispense with. Admitting my ability to conceive infinite space, I may yet doubt its existence,* which doubt I may proceed to remove by proof. I may assume on *à priori* grounds the existence of infinite space, as a postulate of Reason; or I may deny the possibility of establishing the proposition *à priori* and be forced to fall back upon observation, which proceeds by adding space to space. In the latter case, although I may admit that the conception of the infinite is qualitative, and means in the case in point only that a progression along a given line may be

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* I purposely avoid all consideration of metaphysical arguments drawn from theories as to the nature of space.

endless, yet the only apparent possibility of establishing this fact with reference to any given line lies in a continued addition of parts. That is, though the conception itself may be qualitative, and contain no reference to a whole and parts, the only way in which it seems possible to prove any actual object infinite is by employing the ideas of quantity and totality in adding part to part, which ideas are in fact contradictory to ideas already contained in the conception.

Now, it is evident that observation, adding part to part, could never attain the infinite, for the infinite time in which it is said to be possible to accomplish this is no quantity of time at all, and the phrase merely expresses in a disagreeably tautological and roundabout way that it can never be. But in spite of this, since our experience in going from part to part seems to be analogous to what would be our experience of the object if it were, indeed, infinite; and since there seems to be no other way of proving from experience its infinity, we are apt to imagine that by this process we are somehow proving the infinity of an object, or are at least in the way to prove it. Evidently

this feeling was in the mind of Kant when he put forward the first of his antinomies; and it is also evident that it exerted its influence on the mind of Hamilton, leading him to "rise from sphere to sphere in the region of empty space" in the fruitless endeavor to exhaust infinite space. The same feeling it was that caused Clifford to affirm that, when we have counted one, two, three, four miles up into space, although we must admit that the attempt can never give us infinite space, we may yet regard our task as "distinctly begun." The attempt must fail, he thought, but this is the only way even to make the attempt.

Now, if Sir William Hamilton had set before himself the task of proving that we can never *know* space to be infinite, his argument would have been to the point. If he denied the possibility of knowing space to be infinite *à priori*, no method of proving it such was left except the method of observation. He could not, of course, reasonably hold that the empirical method is a true method of proving anything to be infinite, but he might prove with the arguments that he has used for another purpose that the only even apparent

method of proving space infinite is *merely* apparent, and that consequently we can never know space to be infinite.

Or if Sir William had set before himself the task of proving that we cannot imagine or call up before the mind an exact and complete representation of an infinite line, his arguments would not have been aside from the point. The imagination can picture but a small extent of line at any one time; this portion must consequently be limited; each portion successively added in imagination must also have its limits; and we have no escape from the very difficulty which we have before met in the attempt to know an object as infinite,—the impossibility of getting rid of the limits altogether. Whether we conceive our conception of an infinite line to be quantitative or qualitative, there is an equal impossibility of getting rid of the quantitative in trying empirically to prove a line infinite and in imagining an infinite line.

Sir William accepted this argument as a proof of the inconceivability of the infinite. Now, an argument which will prove any infinite object unknowable as an actually existing thing will

prove an infinite object unimaginable only on the supposition that the element which has caused the impossibility in the former case be also present in the latter. An argument which would prove that I cannot know a line to be infinite, because I cannot in the attempt to prove it such get rid of the element of quantity, would prove that I cannot imagine an infinite line, provided that the attempt to imagine such a line must be similar in kind to the former process, and must meet the same difficulty with the quantitative element. And, similarly, an argument which would prove an infinite line unknowable and unimaginable on account of the difficulty occasioned by this quantitative element, would prove an infinite line inconceivable (in the narrower sense of the term) only if the element is necessarily present in conceiving (in the narrower sense) an infinite line that has caused the trouble in the former cases.

But if we can prove that, when the notion of quantity is eliminated from a certain complex of psychic elements, the remaining elements can be in some way grasped in mind, and are thus grasped in certain actual and not infrequent mental operations, in this case the proofs, which

might be suitable when applied to the former propositions, can have no force whatever.

When we come in fact to make a more complete analysis of the elements comprehended in our conception of the infinite, we will see that the arguments which have been so often advanced to prove the infinite inconceivable, and regarded as so unanswerable, are really not to the point at all, but may be classed under the old logical fallacy of *ignoratio elenchi.* Sir William Hamilton proves inconceivable something which is not at all supposed by the phrase "an infinite line," and which, in fact, contains an element flatly contradictory to one of those indicated by the phrase.

In view of the foregoing, the exhibition of the danger in which we stand of confounding three distinct propositions and their appropriate proofs, the two points which I would insist upon are these: (1) That the conceivability of the infinite, in the narrower sense of the term, is quite distinct from a knowledge of its existence, or from its imaginability, and it is with the possibility of the first alone that we have to do; and (2) that, whatever be the metaphysical tenets embraced by one as to the nature of Space and Time, or

of those universal laws of Being known by one school as Rational Intuitions and by an opposing school as Highest Generalizations from Experience,—that these tenets, however they may lead to the acceptance or rejection of the proposition which affirms the existence of an infinite, can yet not affect the proposition which affirms its conceivability.

CHAPTER VI.

THE CONCEIVABILITY OF THE INFINITE.

When we analyze the mental state in which we have reference to an infinite—let us take, for example, an infinite line—we find the following elements: in the first place, there are present the usual qualities of a line; for the fact of our conceiving it as without limits need not alter any of its usual qualities, any more than the fact of our being unable to see the ends of a telegraph wire need force us to deny that it is a wire of a certain diameter, material, or color; and, in the second place, there is present the notion that, however far we may go in thought, we will find a continuation of the line. In other words, there is the notion of *unlimited possibility of quantity*,—a notion which, be it marked, is strictly qualitative. That there is no quantitative element present has been insisted upon in previous chapters. But quantity in general, not this or that quantity, is as much a qualitative notion as color or form;

and in defining the second element present in our conception of an infinite line, I have used the word advisedly to bring out what is a distinctive characteristic of the conception. The word infinite does not denote a quantity, but it has reference to quantity, and it cannot, in accordance with its derivation and true signification, be rightly applied to what is incapable of being quantitatively considered. My objection to the usage of the word infinite, by some who recognize that the conception for which it stands is qualitative, is that they overlook the distinctive characteristic of this conception, which marks it out from other qualitative conceptions,—that is, its necessary reference to quantity, though not itself quantitative. If, by that process of abstraction which takes place when I compare objects similar in some of their qualities, I fix my attention upon the other qualities of any finite line, disregarding its length, and leaving out of view for the time being its limits, my conception is qualitative; and yet it is not the conception of an infinite line. In this case, so far from affirming infinite length, I do not think of length at all. But, in the case of an infinite line, I add to

the former complex of qualities a new quality, possibility of quantity in general, not this or that quantity. When I try to bring before my mind the notion of an infinite line, what I am distinctly conscious of is this: I represent in imagination a line of indefinite length, and then run mentally along the line representing additional line-portions,—a proceeding which would of itself of course give me only the finite; but what makes my conception distinctively of the infinite is that, in this progression or notion of continued increase, I fix my attention upon the progression itself, and eliminate by abstraction the limits to which such a progression is subject. I do not, be it marked, merely fix my attention upon the other qualities of a given line, abstracting from the notion of limits; but I have in mind a progression, a possibility of ever-increasing quantity, and I abstract from the limits of this progression. The two conceptions are distinctly different, although both are qualitative, and they should not be confounded with one another.

The question, therefore, whether I can conceive an infinite line is identical with the question whether I can mentally grasp the various

qualities of a line, and the notion of a continual increase of such a line, without including the notion of limits; and it will be seen that this question is simply one of the phases of the broader question which is concerned with the possibility of the concept or general notion. A certain complex of qualities being necessary to the existence of a given object in nature, or to its subjective existence as represented in the imagination, is there any mental operation by which we may grasp some of these qualities, to the exclusion of others, and convey to our own and other minds by the use of the word which stands for this new complex a distinct meaning?

There have been held with reference to this problem, as is well known, three opinions: the doctrine of the Realists, that general ideas have corresponding to them a counterpart Reality,—a doctrine which may be passed over as now abandoned, though its effects make themselves felt in many directions; the doctrine of the Conceptualists, that, although general ideas cannot exist in Nature, nor be represented in the Imagination, yet they have a true mental existence, and are the result of a distinct men-

tal operation; and the doctrine of the Nominalists, that the only generality that has a separate existence, subjective or objective, is the Name, which may be applied indifferently to many similar objects.

The Conceptualist may hold that it is possible, unless the words include repugnant elements, to conceive an infinite line,—that is, to grasp in mind a certain complex of psychic elements which are yet incapable of being pictured in the imagination as an infinite line. To think, in the sense of to form such a concept, is to him something other than to imagine. What cannot be imagined may yet be thought. The word man, which we define as comprehending the elements of rationality and animality, he claims, does not in the least include all those other qualities which must be combined with these two before we can picture in the imagination or know as existing any given man. If we select the two qualities in which all the objects of a class resemble each other, and give to these two a special name, have we not brought them into consciousness in some way in which we have not the other qualities?

And when we turn to the Nominalist, it would not be hard to show that, although his doctrines, if taken in strictness, would deny the possibility of the mental operation by which we arrive at the concept, and consequently of the operation by which we may grasp in thought the various elements implied in the phrase "an infinite line," yet one may find in his teachings by implication ample justification for assuming its possible and actual existence. I will take some extracts from four well-known Nominalists to show how palpable is the fact stated, and I will first quote from Berkeley, Locke's great opponent on the subject of the Abstract Idea:

"Whether others have this wonderful faculty of abstracting their ideas, they best can tell; for myself, I find, indeed, I have indeed a faculty of imagining or representing to myself the ideas of those particular things I have perceived, and of variously compounding and dividing them. I can imagine a man with two heads, or the upper parts of a man joined to the body of a horse. I can consider the hand, the eye, the nose, each by itself abstracted or separated from the rest of the body. But,

then, whatever hand or eye I imagine, it must have some particular shape and color. Likewise the idea of man that I frame to myself must be either of a white, or a black, or a tawny, a straight, or a crooked, a tall, or a low, or a middle-sized man. I cannot by any effort of thought conceive the abstract idea above described. And it is equally impossible for me to form the abstract idea of motion distinct from the body moving, and which is neither swift nor slow, curvilinear nor rectilinear; and the like may be said of all other abstract general ideas whatsoever. To be plain, I own myself able to abstract in one sense, as when I consider some particular parts or qualities separated from others, with which, though they are united in some object, yet it is possible they may really exist without them. But I deny that I can abstract from one another, or conceive separately, those qualities which it is impossible should exist so separated, or that I can frame a general notion by abstracting from particulars in the manner aforesaid, which last are the two proper acceptations of *abstraction*. And there is ground to think most men will acknowledge themselves to be in my case.

The generality of men, which are simple and illiterate, never pretend to *abstract notions.* It is said they are difficult, and not to be attained without pains and study; we may, therefore, reasonably conclude that if such there be, they are confined only to the learned."*

So much for Berkeley's position with respect to the abstract notion. But mark the concessions which he is forced to make in a later section:

"But here it will be demanded, how we can know any proposition to be true of all particular triangles except we have first seen it demonstrated of the abstract idea of a triangle which equally agrees to all? For, because a property may be demonstrated to agree to some one particular triangle, it will not thence follow that it equally belongs to any other triangle, which in all respects is not the same with it. For example, having demonstrated that the three angles of an isosceles rectangular triangle are equal to two right ones, I cannot, therefore, conclude this affection agrees to all

* "Principles of Human Knowledge." Introduction, Sect. 10. Works, ed. Fraser, vol. i. pp. 141, 142.

other triangles which have neither a right angle nor two equal sides. It seems, therefore, that to be certain this proposition is universally true, we must either make a particular demonstration for every particular triangle, which is impossible, or once for all demonstrate it of the abstract idea of a triangle, in which all the particulars do indifferently partake, and by which they are all equally represented. To which I answer, that though the idea I have in view whilst I make the demonstration be, for instance, that of an isosceles rectangular triangle whose sides are of a determinate length, I may nevertheless be certain it extends to all other rectilinear triangles, of what sort or bigness soever. And that because neither the right angle, nor the equality, nor determinate length of the sides are at all concerned in the demonstration. It is true the diagram I have in view includes all these particulars, but then there is not the least mention made of them in the proof of the proposition. It is not said the three angles are equal to two right ones, because one of them is a right angle, or because the sides comprehending it are of the same length. Which sufficiently shows that the

right angle might have been oblique and the sides unequal, and for all that the demonstration have held good. And for this reason it is that I conclude that to be true of any obliquangular or scalenon which I have demonstrated of a particular right-angled equicrural triangle, and not because I demonstrated the proposition of the abstract idea of a triangle. And here it must be acknowledged that a man may consider a figure merely as triangular without attending to the particular qualities of the angles or relations of the sides. So far he may abstract, but this will never prove that he can frame an abstract, general, inconsistent idea of a triangle. In like manner we may consider Peter so far forth as man, or so far forth as animal, without framing the forementioned abstract idea either of man or of animal, inasmuch as all that is perceived is not considered."*

In the former of the two extracts Berkeley has declared himself able to abstract only so far that he can represent to himself in imagination what can exist separately in nature.

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* "Principles." Introduction, Section 16.

He denies that he can conceive separately those qualities which it is impossible should exist separately. But when he supposes an objector to ask how it is possible for something proved to be true of a particular triangle, to be known to be true of all triangles, he answers that it is seen that neither the right angle, nor the equality, nor the determinate length of the sides are at all concerned in the demonstration. In other words, he admits that, so far as that demonstration goes, we have to do only with those elements in which all triangles agree. And if we can reason about certain elements to the exclusion of others; if we can see that certain objects are alike in certain elements and unlike in the others; if we can give a name to objects simply to express the presence of these same elements, however the elements accompanying them may vary, then surely the elements of the concept have been before the mind in some way in which the others have not, and have been grasped together.

Berkeley frankly admits as much in the concluding sentences of the latter extract, sentences which were added twenty-four years after the first publication of the essay, when

mature reflection, we may suppose, had brought him to see that on his previous principles, strictly held, all comparison of objects differing in any of their qualities would be impossible. If we can consider a figure merely as triangular, without attending to the particular qualities of the angles or relations of the sides, then we can in some sort divorce the elements included under the general word triangle from the accompanying elements and consider them separately. In those last few sentences Berkeley admits all that a reasonable Conceptualist would care to prove, and the words "abstract idea," as there used, are equivalent to "object of the imagination," a something which is not implied in the formation of the abstract or general notion.

Every one, Nominalist or Conceptualist, must acknowledge that we can compare objects and recognize them as like or unlike,—not merely like or unlike as wholes, but in this or that element; like in length, unlike in breadth; like in color, unlike in shape. Now, no one claims that we can call into clear consciousness the element of length alone, and picture it divorced of breadth and color; but when we recognize

two objects as like in length and unlike in breadth, the elements must in some way have been present in mind separately, so as to be recognized as length and breadth. If one object that what is present in consciousness must *ipso facto* be perceived, and that we cannot perceive length as a factor by itself, nor recall in memory any perception of such a factor during the act of comparison, I answer that what is in consciousness is by no means necessarily in a clear analytic consciousness, and that we may by a process of deductive reasoning be sure that certain elements are present as factors in a given mental state, while we are yet quite unable to call these elements into a clear analytic consciousness, separated from certain other elements bound to them by long association and habit. As an instance, I refer to vision. That distance is itself unperceivable by sight we must admit. That judgments of distance are a result of reasoning from an observed constant connection of certain visual with certain other elements, may be satisfactorily established when the above proposition is admitted. But to call into clear consciousness by itself the purely visual sensation, which

forms the basis of the judgment, is altogether impossible. That it is a factor, and an important factor, in the complex consciousness which we have at the time, we know, and yet its presence, as a single and distinct element, is capable of being only deductively known. Notice a further point which is worthy of remark. If we vary the purely visual element, allowing all the other elements to remain the same,—that is, if we change the *color* of the object, but do not change in any respect the form or size of the image on the retina,—a difference is at once remarked, and the change of color recognized. But the difference is not recognized as a difference between two purely visual sensations when the result of the actual comparison comes into clear consciousness, but as a difference in one of their elements between two complexes or wholes. That is to say, the two visual sensations cannot be brought into clear consciousness and compared with each other alone, but only as they are connected with certain other elements in complexes or wholes; it is the presence of two or more such wholes, which we wish to compare, that primarily impels to the narrowing of the atten-

tion to the single similar or dissimilar elements. This point is specially worthy of remark, as there is something closely analogous to this in the formation of the concept in general, and this special case may help to throw light upon all cases in which that which cannot be imagined is yet thought.

When I form the concept of length by comparing two objects in length and affirming agreement, and then recognizing as a distinct element that in which they agree, I certainly do not compare the objects simply as wholes, but compare the *lengths;* and I must surely have had these elements in mind in some way in which I had not the other elements which go to make up the object. Whether I can call into clear consciousness the psychic elements present during the operation or not, it does not much matter. I evidently have specialized, selected some elements from among others, and compared length with length, as element with element. The name which we give to such resemblances is the name representing a general or abstract idea. Whether the possibility of thus comparing single elements may not be always conditioned by the presence of two or

more objects or complexes in which the elements are present I will consider later.

Hume warmly applauds the position taken by Berkeley with reference to the abstract idea, calling it "one of the greatest and most valuable discoveries that have been made of late years in the republic of letters," and he undertakes to confirm it with proofs that he hopes will put it "beyond all doubt and controversy." For the same purpose for which I quoted the two extracts from Berkeley, I will quote the last part of the section which he devotes to the establishment of this position:

"It is certain that the mind would never have dreamed of distinguishing a figure from the body figured, as being in reality neither distinguishable, nor different, nor separable, did it not observe that even in this simplicity there might be contained many different resemblances and relations. Thus, when a globe of white marble is presented, we receive only the impression of a white color disposed in a certain form, nor are we able to separate and distinguish the color from the form. But observing afterwards a globe of black marble and a cube

of white, and comparing them with our former object, we find two separate resemblances in what formerly seemed, and really is, perfectly inseparable. After a little more practice of this kind we begin to distinguish the figure from the color by a *distinction of reason*,—that is, we consider the figure and color together, since they are, in effect, the same and undistinguishable, but still view them in different aspects, according to the resemblances of which they are susceptible. When we would consider only the figure of the globe of white marble, we form in reality an idea both of the figure and color, but tacitly carry our eye to its resemblance with the globe of black marble; and in the same manner, when we would consider its color only, we turn our view to its resemblance with the cube of white marble. By this means we accompany our ideas with a kind of reflection, of which custom renders us, in a great measure, insensible. A person who desires us to consider the figure of a globe of white marble without thinking on its color, desires an impossibility; but his meaning is that we should consider the color and figure together, but still keep in our eye the resemblance to

the globe of black marble, or that to any other globe of whatever color or substance."*

It is not hard to see that we cannot distinguish in a body figured "many different resemblances and relations" without bringing the resembling elements in some sense singly into thought; if the mental complex which we call an object were an indissoluble unit, we could affirm a general likeness or unlikeness between it and other objects, but we could not affirm that the resemblance lay in the figure or color. If, as Hume asserts, the figure and color "are, in effect, the same and undistinguishable," why do we find the one susceptible of the one class of resemblances and the other of another class? If we take the words literally, should not the figure, viewed in one aspect, be susceptible of resemblances of figure, and viewed in another of color? And, similarly, if the color is one with the figure,—the same and undistinguishable,—should not the color, viewed in one aspect, be susceptible of resemblances of color, and viewed in another of figure? Hume's admis-

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* "Treatise of Human Nature," bk. i. Sect. 7. Works, Boston, 1854, vol. i. p. 42.

sion that the two elements are known as giving different resemblances, in itself refutes his previous assertion that they are undistinguishable. If color be recognized as like color, and figure like figure, the two qualities are distinguished as different, and are in reality separately grasped.

I will now take a passage from Mr. J. S. Mill's "Examination of Sir William Hamilton's Philosophy":

"The formation, therefore, of a Concept does not consist in separating the attributes which are said to compose it from all other attributes of the same object, and enabling us to conceive those attributes, disjoined from any others. We neither conceive them, nor think them, nor cognize them in any way as a thing apart, but solely as forming, in combination with numerous other attributes, the idea of an individual object. But, though thinking them only as part of a larger agglomeration, we have the power of fixing our attention on them to the neglect of the other attributes with which we think them combined. While the concentration of attention actually lasts, if it is sufficiently intense, we may be tempo-

rarily unconscious of any of the other attributes, and may really, for a brief interval, have nothing present to our mind but the attributes constituent of the concept. In general, however, the attention is not so completely exclusive as this; it leaves room in consciousness for other elements of the concrete idea; though of these the consciousness is faint in proportion to the energy of the concentrative effort, and the moment the attention relaxes, if the same concrete idea continues to be contemplated, its other constituents come out into consciousness. General concepts, therefore, we have, properly speaking, none; we have only complex ideas of objects in the concrete; but we are able to attend exclusively to certain parts of the concrete idea; and by that exclusive attention we enable those parts to determine exclusively the course of our thoughts as subsequently called up by association, and are in a condition to carry on a train of meditation or reasoning relating to those parts only, exactly as if we were able to conceive them separately from the rest." *

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* "Examination of Sir William Hamilton's Philosophy," vol. ii. p. 64. Boston, 1868.

This passage is so clearly in harmony with the views of the Conceptualist, as I have portrayed them, that it seems scarcely necessary to comment upon it. But I cannot resist the temptation to delay for a moment over an inconsistency into which Mill was forced by his attempt to recognize, though a Nominalist, a truth which the Nominalist, pure and simple, cannot recognize. The formation of a Concept, he insists, does not consist in "separating the attributes said to compose it from all other attributes of the same object, and enabling us to conceive those attributes, disjoined from any others." This position he emphasizes by the further affirmation that "we neither conceive them, nor think them, nor cognize them in any way as a thing apart, but solely as forming, in combination with numerous other attributes, the idea of an individual object." These sentences are certainly unequivocal: they contain an emphatic assertion of the nominalistic doctrine.

But, side by side with such statements, we find it asserted that we may fix the attention upon the attributes constituent of the concept, to the neglect of the other attributes of the

object, and that while the concentration of attention actually lasts, if it is sufficiently intense, "we may be temporarily unconscious of any of the other attributes, and may really, for a brief interval, have nothing present to our mind but the attributes constituent of the concept." Surely, if the only elements before the mind are those constituent of the concept; if we may be conscious of these, even for a brief interval, and conscious of these alone; surely in such a case we conceive, or think, or in some way cognize the attributes forming the concept as separate and apart, and *not* for the time being, in combination with numerous other attributes. Mr. Mill goes even further in the above admission than most of us would care to follow him. In speaking as he does of the process, and not distinguishing between the imagining of an object and the knowing of one or more of its isolated qualities, he clearly intimates, although he does not distinctly say, that the elements before the mind during the formation or use of a concept are in consciousness in the same way that the whole complex or object may be in consciousness. But, to recur to the before-mentioned analogy of the purely

visual element in vision, we know that, although we may so concentrate the attention as to distinguish the blue color of one object from the red color of another, and so must have compared in some rapid manner these purely visual sensations, yet when we try to call into clear consciousness the mere sensation of color, we cannot do it without imagining the color as on a surface, or combined with psychic elements not purely visual. That is to say, the single and separate sensations cannot be called into a clear consciousness, and their presence when we use the concept, or have occasion to compare them singly with each other, is something quite distinct and different from the presence in consciousness of the complex which is knowable as an object. And such would seem also to be the case wherever we call before the mind the single psychic elements which can yet not be represented alone in the imagination. The element must have been grasped separately, but it can be brought into a clear consciousness only in combination.

If now we recognize in each of two objects presented to us a certain quality or complex of qualities upon which we can fix the atten-

tion, and if we discover that, so far as these qualities go, there is an undistinguishable similarity in the objects, the differences arising altogether from other qualities, why may we not call the complex of qualities in point a *general notion* or *general idea?* Of course, whether we should call the qualities in the two instances the *same*, even if they were undistinguishably similar, would depend on our use of the word same,* and our ideas of what constitutes sameness or identity; but I can see no objection to using the words "general notion" to indicate the fact that a certain complex of qualities is to be found in many different combinations with other qualities. Should it still be insisted that, since we cannot bring separately into clear consciousness these elements of objects known, we have no reason to assume that we actually conceive them or think them separately, I will not quarrel over the use of a word, but will simply state that I find the word "conceive" a useful one to express that concentration of the attention upon certain

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* I have pointed out before the fact that the word 'same" is commonly used in four quite distinct senses.

qualities of an object which takes place when objects are compared, and which eliminates from consciousness, or at least subordinates all other qualities of the objects; and I will so use the word, applying it to an operation the existence of which Mr. Mill has in so many words admitted.

The last author whom I will quote is Mr. Bain. I will take some passages from the chapter on abstraction in his compendium of psychology and ethics, where he supports the Nominalistic doctrine:

"We are able to attend to the points of agreement of resembling things, and to neglect the points of difference, as when we think of the light of luminous bodies, or the roundness of round bodies. This power is named Abstraction.

"It is a fact that we can direct our attention or our thoughts to the points of agreement of bodies that agree. We can think of the light of the heavenly bodies, and make assertions, and draw inferences respecting it. So we can think of the roundness of spherical bodies, and discard the consideration of their color and size. In such an object as the full

moon we can concentrate our regards upon its luminous character, wherein it agrees with one class of objects, or upon its figure, wherein it agrees with another class of objects. We can think of the taste of a strawberry, either as agreeing with other tastes or as agreeing with pleasures generally. . . ."

"Every Concrete thing falls into as many classes as it has attributes; to refer it to one of these classes, and to think of the corresponding attribute, are one mental operation.

"When a concrete thing before the view recalls others agreeing in a certain point, our attention is awake upon that point; when the moon recalls other luminous bodies, we are thinking of its light; when it recalls other round bodies, we are thinking of its roundness. The two operations are not different but identical.

"On this supposition, to abstract, or to think of a property in the abstract, is to classify under some one head. To abstract the property of transparency from water is to recall, at the instance of water, window-glass, crystal, air, &c.; to abstract its liquidity is to recall milk, vinegar, melted butter, mercury, &c.; to

abstract its weight is to bring it into comparison with other kinds of gravitating matter.

"Hence abstraction does not properly consist in the mental separation of one property of a thing from the other properties, as in thinking of the roundness of the moon apart from its luminosity and apparent magnitude. Such a separation is impracticable; no one can think of a circle without color and a definite size. All the purposes of the abstract idea are served by conceiving a concrete thing in company with others resembling it in the attribute in question, and by affirming nothing of the one concrete but what is true of all those others."

. . . "In abstract reasoning, therefore, we are not so much engaged with any single thing as with a class of things. When we are discussing government, we commonly have in view a number of governments alternately thought of; if we notice in any one government a certain feature, we run over the rest in our mind, to see if the same feature is present in all. There is no such thing as an idea of government in the abstract; there is only possible a comparison of governments in

the concrete; the abstraction is the likeness or community of the individuals."*

It will be noticed that throughout this extract Mr. Bain does not distinguish between the elements of an act which come out into a clear consciousness and the elements which do not so come out, but are necessary to the possibility of the operation. When Mr. Bain says, for instance, that there is no such thing as the idea of government in the abstract, but that we can compare governments in the concrete, and recognize the likeness of the individuals, it is perfectly true that all that we are clearly conscious of is several individual objects and a similarity between them; but when we come to analyze this recognition of a similarity, it will be seen that the elements which are known as similar are quite incapable, by themselves, of forming a concrete object, and yet they are distinguished by the mind from the dissimilar elements; they must, therefore, have been in some sort grasped separately, though they cannot separately be brought into a clear consciousness. Whether, during this rapid act of

* "Mental and Moral Science," London, 1868, pp. 176–78.

concentration and comparison, the other elements which go to form the object actually disappear from consciousness, or are only dimly perceived, as Mr. Mill suggests that they are in most cases, does not affect the peculiar character of the act. When I compare in height two trees which I see side by side in the distant landscape before me, I am perhaps conscious of several objects in their immediate vicinity in a dim and indefinite way, but the two objects compared are present in consciousness in a manner very different, and are grasped, so to speak, separately. And when I fix my attention upon the height of the two trees, finding them similar or dissimilar in this one element, we have every reason to suppose that something very analogous takes place, and that this one element is present in consciousness in some way quite different from the others, and is grasped separately for the time being. Were it not so, we could not say the trees are alike in height, but different in contour or color of foliage. We are justified in assuming that when we recognize two trees as like in height, but not in color, we have compared height with height and color

with color, and not merely compared the one object as an undistinguishable complex of qualities with another object as another undistinguishable complex.

As I have before said, the name which we choose to apply to this operation is of little consequence; the point chiefly to be borne in mind is that we have here an operation differing from ordinary imagination, in that it takes cognizance of certain psychic elements which can yet not be called into clear consciousness by themselves as a mental picture. Whether the two operations completely differ in their ultimate nature is another question. When the Conceptualist asserts that though he cannot imagine length apart from breadth or color, yet he can conceive or think it, he merely marks by a distinct name his recognition of an operation different from imagination, and which is implied in all comparison of objects. What may be the peculiar psychic elements present in the operation he does not necessarily know, nor express when he uses the name.*

* Kant seems to have despaired of the possibility of ever making this analysis: "Dieser Schematismus unseres Ver-

Arguing from the analogy of the purely visual element in vision, one might conclude that what is actually present in consciousness in comparing lengths, for instance, is the distinctive element which is present in combination with other elements (and, consequently, in a modified form) in all our experience of extended objects, but which, in the act of comparing two objects, may be brought into sufficient prominence to be considered, for the moment, alone, and alone compared with its kind. When we make the attempt to call it into clear consciousness, the element appears as modified by, and in combination with, others; but it is not improbable that, in the act of comparison, it obtains in its pure state sufficient recognition to make possible a comparison with a similar element also in its pure state. However, whether we can describe just what

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standes, in Ansehung der Erscheinungen und ihrer blossen Form, ist eine verborgene Kunst in den Tiefen der menschlichen Seele, deren wahre Handgriffe wir der Natur schwerlich jemals abrathen und sie unverdeckt vor Augen legen werden" ("Kritik, von dem Schematismus der reinen Verstandesbegriffe"), but he did not doubt the operation.

is present during the act or not, we may be sure that a mental separation of two objects into their elements is necessary in order to a recognition of them as in some points similar and in some dissimilar.

Hume has asserted that if we knew but the one object, and had no other objects with which to compare it, we would never distinguish between the several elements composing it; and the same thought is made prominent by Mr. Bain when he states that to refer an object to a class of other objects, and to think of the corresponding attribute, are but one act. Mr. Bain's statement is, of course, somewhat inaccurate, as it confounds two very different parts of the one operation; but the point upon which both of these writers insist,—namely, that a comparison of objects is necessary to an analysis of the objects into their resemblances,—what should be called their ultimate elements,—is worthy of attention. In all probability, were it not for a comparison of objects, a constant experience of groups of psychic elements containing likenesses and unlikenesses, we should never analyze the groups and make prominent single elements, separated

from their accompaniments. And since the single elements do not themselves come out into a clear analytic consciousness, it is not easy to see how we could be sure that they had been separately grasped, if we could not infer their presence from the possibility of comparison and classification of objects. That this analysis implies the presence of two objects, is necessarily a classification, after the analytic habit has been formed, as Mr. Bain insists, is not so clear; but that, as a preliminary to the act of concentration by which we form the concept, we call into consciousness at least one concrete object, I think, cannot be doubted; and this fact might easily mislead one into taking the Nominalistic position that a recognition of particular objects expresses the whole process. Particulars must be present of course, if they are to be compared; but, when compared, they are not compared as wholes.

In view of the foregoing, I would, therefore, regard the fact as beyond all doubt, that there are mental operations differing distinctly from imagination, in that certain elements, of which we have usually, as single elements, no analytic consciousness, but which are merged with

others into an indivisible whole, are brought for the time being into such prominence as to be compared individually with similar elements, and recognized as like or unlike. It does not follow that we may have a clear consciousness of the steps in the rapid process in which this comparison takes place, or clearly recognize the nature of the elements compared; but from the fact of the comparison, about which there can be no doubt, we may be very sure that the operation in question has taken place.

Now, when we return to the particular conception which we have been considering,—that of an infinite line,—we find it merely a concrete instance of this general truth, which all must either explicitly or implicitly admit. As I have said, the elements constituent of this conception are the usual qualitative attributes of a line and the notion of continued progression, of unlimited possibility of quantity. These elements may be brought into mind, to the exclusion of the notion of limits, which are yet present in all imagined lines and in all intuitions of lines in nature, by employing the process usual in forming a concept. When I think of an infinite line, I first represent to

myself a line of some indefinite length, and I then run mentally along this line, adding new portions,—that is, I successively think several increasing lengths. I have now before my mind what Hume and Bain insist upon and make so prominent in forming the concept, several concrete objects similar in some of their qualities. Having mentally passed over several of these line portions, I then fix my attention, not upon that in which they differ,—the quantitative element,—but upon that in which they resemble, the usual qualitative attributes of a line, and the notion of increase or progression, which is common to all. This is precisely what I do in forming the concept man or animal. The concrete objects are compared, their differences eliminated by abstraction, and their likenesses grasped together under a distinctive name. Or I may select one of the qualities in which objects agree, and consider it alone, as when I compare men of the same age and color, only as to their height, and pronounce them equal in height. If this be possible, if, in using the word man, I can distinguish between that in which men agree and that in which they disa-

gree, and if it be further possible for me to fix my attention upon one of the points in which they agree, to the exclusion of others, then it is possible to abstract from the particular quantities or amounts of several lines present in imagination, and think only of a constant increase or progression. That both the one and the other are not only possible but actual operations, is proved beyond possibility of doubt by our constant comparison of objects, our use of general language, our frequent use of the word infinite, to indicate what is clearly distinguished, readily defined, and conveys a distinct meaning to speaker and hearer.

One point I will here remark upon before passing, and that is the distinction sometimes drawn between the abstract and the general notion, a point at which I have a few pages before briefly hinted. Admitting that Hume was right in saying that had we not had occasion to compare two objects we should never have analyzed either into its elements, the question naturally arises, whether, after we have formed this habit of analysis by comparison, we may not by mere effort of will fix the attention upon one element of a single object, without any

reference to its occurrence in another object? To take an example, can I not, in imagining a window or a door, fix my attention upon its length, without thinking of the length of anything else, or comparing the object with any other extended object?

The question cannot be answered off-hand, as by saying that in recognizing my perception as a perception of length—in using the word length—I necessarily class the object with other long objects; for it is at least thinkable that I may have so associated the word with this peculiar element as to have it suggested by the presence of the element, and still may not be conscious of other combinations in which the element occurs. It is, I think, highly probable, however, that when we concentrate attention upon one element of an object, there is a more or less dim and vague reference to other objects, and that there is a rapid comparison; but this fact must be proved by an interrogation of consciousness during the act, and upon this point I will not insist. If it be allowed that the one element may be known without reference to its occurrence in two or more objects, we have what may justly

be called the abstract notion, as distinguished from the element recognized as present in several combinations, in which latter case we may call it the general notion. And even if we deny that the abstraction is possible except there be two or more objects present in mind, and a comparison of them, yet it must be acknowledged that the prominence of these objects in consciousness varies greatly, and, accordingly, we may have either the intension or the extension of the concept most prominently before the mind. If we are concerned, not so much with the combinations in which the element occurs as with the element itself, we may call our notion abstract; if we have prominently in mind the number of occurrences, we may call it general. In either case the distinction between the abstract and the general notion is a legitimate one.

This point is not one directly connected with the subject with which I am concerned, —the conceivability of the infinite,—but is one too interesting to be overlooked in any examination into the nature of the concept; and, indeed, it can scarcely be considered totally foreign to the subject in hand, as the operation

of forming a concept, and the act of conceiving an infinite, are not different in their nature, and may be viewed in the same aspects. And to the objection which may be made to my classing the notion of this or that particular infinite line with the concept or general notion, as I have done throughout this chapter, the objector taking the ground that the individual or the intuition is something quite different and distinct from the concept,—to this objection I answer that the notion of any particular infinite line is not a complete intuition, in that one of the elements of the intuition is eliminated by abstraction; and that when, in the formation of any concept, we fix the attention upon certain elements of an intuition to the exclusion of others, we have in mind, so to speak, a constituent part of an intuition: the fact that we recognize its similarity, or, if we so choose to use the word, its sameness with parts of other intuitions, does not alter the individual character of the elements which we actually have in mind. The operation of forming a concept and the operation of conceiving an infinite line are in nature identical.

It seems impossible that any one, having

reflected upon the fact of his constantly grasping in concepts elements which can yet not be separately imagined, and having, after an analysis of what is in his mind when he calls up the notion of an infinite, discerned the identity of this latter operation with the former, it seems impossible that such an one should hold an infinite line, or infinite time or space to be inconceivable. But, being loath to give up his former position, such a man will probably put forward in a new form an objection upon which I have already commented. We have heard him object, "If we do not know the infinite as a whole, do we not know only its parts, which are finite?" And now we will hear him object that, "Even if it be true that we can grasp in thought the notion of progression, and the notion of a line in general, this will give us no knowledge of an infinite line, but will give us only the elements of an incomplete image, which cannot be called distinctly before consciousness, and therefore cannot be known as an object at all." If, however, one feel himself aggrieved because he cannot represent to himself, endowed with all the qualities necessary to an object of the imagination, that which

he has already defined as wanting some of those qualities, he will be unreasonable enough to think it ground for complaint that he cannot in thought make parallel lines meet, or imagine a triangle with four sides. The word infinite means devoid of limits, and it necessarily follows that an infinite line cannot be known as a quantity, consequently not as a whole. Every object which is seen or imagined has necessarily limits, definite or indefinite: an infinite line, as infinite, cannot become an object of the imagination. But from this it by no means follows that I cannot call a particular line infinite, provided I have some proof of the fact other than its conceivability, and that I cannot know my conception to be in harmony with the reality. Suppose that, either from testimony or by means of some *à priori* chain of reasoning, I have good reason to believe a given line endless, I can conceive the line as without end, and I may know my conception, although it does not represent the total content of my consciousness when at any moment I gaze upon this or that part of the line, to be a true and real conception, and in harmony with my experience as I progres-

sively pass over the line; and I may be certain that, however long my experience may continue, it will yet not prove incompatible with the conception I have formed. In this sense, and in this sense alone, is any infinite object conceivable, and there is no other conceivable way in which we could conceive it. An infinite object which could be known as a whole is not even an object of thought, for the elements indicated by the words cannot be so put together as to express a meaning. But the conception of the infinite, as I have defined it, contains in it nothing either contradictory or beyond the grasp of the human mind, and is, indeed, a very common conception, as is evidenced by use of the word infinite in literature, ancient and modern, to say nothing of the constant occurrence of the word in the debates of those very philosophers who find the conception such a stone of stumbling. And that the conception is a real one, having a real consonance with experience, those of us who hold to the Christian doctrine of Immortality will not be slow to maintain.

THE END.